The FENE model for viscoelastic thin film flows: Justification of new models and applications.

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Abstract

In this article, we rigorously determine an asymptotic model for viscoelastic flows of FENE type for thin domains. The proof presented here is based on existence and unicity results for a Fokker-Planck equation and for the limit problem when the ratio between height and width of the physical domain vanished. We finally show that the error between complete FENE constitutive law and the approximation suggested for thin films domains can be controlled. Some applications are given at the end of this article: in the fields of lubrication, phenomena of boundary layers, of the industry of the nanotechnology, of biology or Shallow-Water equations.

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1 Introduction

Motivations - A constitutive law gives a relation between the rate of deformation and the constraint. Simplest of these laws are obtained in an empirical way using comparison with experiments. Certain laws are also justified by analogies with mechanical laws of macroscopic models. Other laws, more recent, utilize microscopic behaviors. In all these cases, we can raise the question to know if being given a velocity field $\mathbf{u}$ of a fluid, is simple to find the constraint $\boldsymbol{\sigma}$? More precisely, the first object of the present paper is to recall that the classical constitutive laws, and in particular the multi-scale FENE law, are well posed, i.e. that there exists a unique constraint for each flow velocity. Once checked this essential mathematical property, we are interested in the behavior of this law in a particular geometry which occurs in very many mechanisms: in thin flows. The applications are numerous and we can legitimately put the following question: can't the law obtained empirically for a fluid be written in a simpler way when the fluid considered lives in a particular geometry? Can we, in this case, deduce an explicit expression for the constraint according to the velocity of the fluid? The answers to these questions are well-known in the case of the Newtonian fluids: for example in the field of lubrication, the Navier-Stokes equations can be rigorously approximate by the Reynolds equation. For a viscoelastic fluid like Oldroyd-B fluid, recent results show that in a thin film the flow is managed by more simple equations (in particular from a numerical point of view) than the Navier-Stokes-Oldroyd models usually obtained.

Mathematical formulations - Generally, the equations describing the hydrodynamics for incompressible fluid are the following conservation laws:

$$\mu \frac{\partial \mathbf{u}}{\partial t} + \mu \, \mathrm{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathrm{div}(\boldsymbol{\sigma}) \quad \text{and} \quad \mathrm{div}(\mathbf{u}) = 0 \qquad \text{on } \mathbb{R}^+ \times \Omega. \tag{1}$$

where $\mathbf{u}(t, \mathbf{x})$ is the velocity at time $t \in \mathbb{R}^+$ and at position $\mathbf{x} \in \Omega$, Ω being a bounded domain in $\mathbb{R}^d$, ($d \in \{2, 3\}$), μ is the density of the fluid, $\boldsymbol{\sigma}$ corresponds to the stress tensor and p corresponds to the pressure. To complete the mathematical formulation of the balance laws, we need a constitutive law relating the stress tensor $\boldsymbol{\sigma}$ to the motion, for instance to the velocity $\mathbf{u}$.

1) The constitutive law for a incompressible Newtonian fluid is given by

$$\boldsymbol{\sigma} = \eta \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right).$$

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Here η is a constant ($\eta > 0$) known as the viscosity and it is clear that for each velocity field $\mathbf{u} \in W^{1,\infty}(\Omega)$ such a stress $\boldsymbol{\sigma}$ is well defined and we have $\boldsymbol{\sigma} \in L^{\infty}(\Omega)$.

2) There are many ways to generalize this linear Newtonian model by inclusion of nonlinear terms. For instance, the so-called generalized Newtonian fluid for which the extra stress is explicitly given with respect to the velocity by

$$\boldsymbol{\sigma} = \eta(|\nabla \mathbf{u}|)\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right),$$

where η is a function. According to this function, we obtain for example the power-law model: $\eta(x) = mx^{n-1}$, the Yasuda-Carreau model given by $\frac{\eta(x)-\eta_\infty}{\eta_0-\eta_\infty} = (1+(\lambda x)^a)^{(n-1)/a}$ where m, n, λ and a are constants determined by experiments. There again, it is clear that for each velocity field $\mathbf{u} \in W^{1,\infty}(\Omega)$ we have $\boldsymbol{\sigma} \in L^{\infty}(\Omega)$ (under condition of course that the function η is sufficiently regular, which is the case for classical generalized Newtonian models).

3) In viscoelastic fluids, the stress does not only depend on the current motion of the fluid, but also on the history of the motion. Such a behavior can be obtained by macroscopic considerations. A classical way to introduce the rheological properties of such a viscoelastic fluid is to compare any elementary fluid element to a mono-dimensional mechanical system composed by springs and dash-pots (see [18]). For example, the UCM Maxwell model is given by this constitutive law

$$\lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\sigma} - (\nabla\mathbf{u})\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\nabla\mathbf{u})^T\right) + \boldsymbol{\sigma} = \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right). \tag{2}$$

The quantity λ has the dimension of time and is known as a relaxation time. It is, roughly speaking, a measure of the time for which the fluid remembers the flow history. Popular models of this type are then obtained including the Giesekus model, which adds a term proportional to $\boldsymbol{\sigma}^2$ to the left side of (2), the Phan-Thien-Tanner model, which adds a term proportional to $\boldsymbol{\sigma}\mathrm{Tr}(\boldsymbol{\sigma})$... In all these cases, we can show that for a regular given velocity field $\mathbf{u}$, the stress tensor $\boldsymbol{\sigma}$ is well defined (that is exists and is unique), see for instance Chupin [11], Guillopé-Saut [17], Renardy [34]. For instance, if $\mathbf{u} \in \mathcal{C}(0,+\infty; W^{1,\infty}(\Omega))$ then the solution $\boldsymbol{\sigma}$ of equation (2) belongs to $\mathcal{C}(0,+\infty; L^{\infty}(\Omega))$.

4) An other way to model viscoelastic behavior is to use microscopic considerations. A kinetic theory corresponding to a diluted solution of polymeric liquids also gives some "constitutive equations" relating the stress tensor $\boldsymbol{\sigma}$ to the velocity field $\mathbf{u}$. The most famous model is the FENE model (Finite Extendible Nonlinear Elasticity) in which a spring tension contribution and a bead motion contribution are added to the Newtonian stress and whose the sum is given by (k and θ are two physical constants which will be presented later)

$$\boldsymbol{\sigma}(t,\mathbf{x}) = \int_{B(0,Q_0)} \mathbf{F}(\mathbf{Q})\otimes\mathbf{Q}\psi(t,\mathbf{x},\mathbf{Q})d\mathbf{Q} - k\theta\left(\int_{B(0,Q_0)}\psi(t,\mathbf{x},\mathbf{Q})d\mathbf{Q}\right)\mathbf{Id} \tag{3}$$

where $B(0,Q_0)$ is the open ball of $\mathbb{R}^3$ centered at 0 of radius Q_0 and $\mathbf{F}$ is the function on $B(0,Q_0)$ defined by $\mathbf{F}(\mathbf{Q}) = \frac{H\mathbf{Q}}{1-\|\mathbf{Q}\|^2/Q_0^2}$ and where the function ψ satisfies the following Fokker-Planck equation (the physical parameter ζ will be presented later too)

$$\frac{\partial \psi}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi = -\,\mathrm{div}_{\mathbf{Q}}\left((\nabla_{\mathbf{x}}\mathbf{u})^T\cdot\mathbf{Q}\psi - \frac{2}{\zeta}\mathbf{F}(\mathbf{Q})\psi - \frac{2k\theta}{\zeta}\nabla_{\mathbf{Q}}\psi\right). \tag{4}$$

Note that for this FENE model, works of Bird and al. [7] (see also part 2.3 of this paper) describe the stress behavior $\boldsymbol{\sigma}$ in a stationary state and for a homogeneous and small velocity flow. This behavior was found more recently by P. Degond, M. Lemou and M. Picasso [13].

Remark 1.1

- These various models of viscoelastic fluids are more or less contained one in another. Thus, taking $\mathbf{F}(\mathbf{Q}) = H\mathbf{Q}$, $H \in \mathbb{R}_+^*$ *in the Fokker-Planck equation* (4) *it is possible to recover the UCM Maxwell model* (2) *for the stress tensor* $\boldsymbol{\sigma}$ *given by* (3). *In the same way taking* $\lambda = 0$ *in the UCM Maxwell model* (2) *we obtain a Newtonian fluid. The model* (3)-(4) *is then the most general.*

- We can also see this hierarchy of models like an increasingly precise description of the flows: from the macroscopic global description of a Newtonian flow whose linear law was discovered about 1687 by Newton to the microscopic description of recent models.

It is important to observe that for any function $\mathbf{F}$, the equation (4) can be writen in a non-dimensional form (i.e. introducing the Deborah number $\mathcal{D}e$, see the next part) as

$$\frac{\partial \psi}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi - \frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) + \operatorname{div}_{\mathbf{Q}}\left((\nabla_{\mathbf{x}}\mathbf{u})^T\cdot\mathbf{Q})\psi\right) = 0 \tag{5}$$

where M is a Maxwellian function depending on $\mathbf{F}$.

Main steps of this paper - The first goal of this paper is to give a rigorous proof that for each velocity field $\mathbf{u}$ there exists a unique constraint $\boldsymbol{\sigma}$ to the FENE model, and then to show that for an almost laminar flow, the behavior of this solution corresponds to the stationary solutions describe by Bird and al. [7]. This article is composed of the following parts:

1. *Description of the FENE model.* We first write the FENE model with its physical parameters then we put it in a non-dimensional form introducing the Deborah (or Weissenberg) number $\mathcal{D}e$. We also indicate how to write the Fokker-Planck equation (4) into the form of the equation (5) and we give some cases where an exact expression of the solution is known.

2. *Existence and uniqueness* for a solution $\psi(\mathbf{Q})$ to an elliptic partial differential equation

$$-\frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) + \operatorname{div}_{\mathbf{Q}}\left(\psi\boldsymbol{\kappa}(\mathbf{Q})\right) = f(\mathbf{Q}) \tag{6}$$

   on a ball $B = B(0,Q_0) \subset \mathbb{R}^d$ and satisfying $\int_B \psi(\mathbf{Q})d\mathbf{Q} = \rho$ where $\rho \in \mathbb{R}$, $\boldsymbol{\kappa}$ is a bounded application from B to $\mathbb{R}^d$ and M is a smooth function from $\overline{B}$ to $\mathbb{R}$ satisfying $0 < M \leq 1$ on B, $M = 0$ on ∂B and $\int_B M = 1$. The two main difficulties come from the fact that the function M cancels on ∂B and that the function V does not make it possible for the operator to be coercitiv.

   This study permits us to affirm that the FENE model admits a solution for a non homogeneous flow in a stationary case.

3. *Existence, uniqueness and long time behavior* for a solution $\psi(t,\mathbf{x},\mathbf{Q})$ to a parabolic partial differential equation

$$\frac{\partial \psi}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi - \frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) + \operatorname{div}_{\mathbf{Q}}\left(\psi\boldsymbol{\kappa}\right) = f(\mathbf{Q}) \tag{7}$$

   for $(t,\mathbf{x},\mathbf{Q}) \in \mathbb{R}_+^* \times \Omega \times B$ whose the initial condition ψ_{init} is given. In this model, $\mathbf{u}(t,\mathbf{x})$ and $\boldsymbol{\kappa}(t,\mathbf{x},\mathbf{Q})$ are two given functions. According to this study we show that at each velocity $\mathbf{u}$ corresponds a unique constraint $\boldsymbol{\sigma}$ for the FENE model.

4. *Asymptotic behavior* for the solutions $\psi^\varepsilon(t,\mathbf{x},\mathbf{Q})$ to

$$\varepsilon\left(\frac{\partial \psi^\varepsilon}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi^\varepsilon\right) - \frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi^\varepsilon}{M(\mathbf{Q})}\right)\right) + \operatorname{div}_{\mathbf{Q}}\left(\psi^\varepsilon\left(\boldsymbol{\kappa} + \varepsilon\widetilde{\boldsymbol{\kappa}}\right)\right) = 0 \tag{8}$$

   when the parameter ε goes to 0. Roughly speaking, we show that the limit of ψ^ε corresponds to the value of ψ^0 (obtained for $\varepsilon = 0$) modulo a boundary layer in time, i.e. except for a correction function depending on t/ε.

5. *Applications* to lubrication and to spatial boundary layers: in thin flows, a model of the type FENE can be put in a non-dimensional form and reveals a small parameter ε (typically the ratio between height of the field and its length in the case of lubrication problem, or the thickness of boundary layer in problem of spatial boundary layer). The model obtained corresponds to the one described by equation (8) and consequently converges to a stationary model described by the equation (6) when ε tends to 0. The approximation suggested by Bird and al. [7] in the case of the steady homogeneous flows is thus usable and makes it possible to obtain relatively simple limiting models.

2 The FENE model

2.1 The Fokker-Planck equation

The simplest non-linear kinetic theory model of a dilute polymer solution is known as the Finitely Extensible Non-linear Elastic (FENE) dumbbell model (see the book of Bird and al. [7]). The polymer solution is viewed as a flowing suspension of dumbbells that do not interact with each other and are convected by the Newtonian solvent. Each dumbbell consists of two identical Brownian beads connected by an entropic spring; see figure 1.

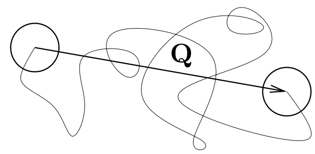


Figure 1: The polymer (in thin line) is modelled by a "dumbbell": two beads linked by a spring.

On the microscopic level, the kinetic theory gives a Fokker-Planck equation for the probability density $\psi(t, \mathbf{x}, \mathbf{Q})$ where $\mathbf{Q}$ denotes the set of variables defining the coarse-grained micro-structure [7, 14]. As presented above, in this paper, we discuss an even coarser model of the single dumbbell, namely two beads connected by an elastic spring. In this case, the configuration variable $\mathbf{Q}$ simply represents the vector connecting the two beads of the dumbbell. In this case, the Fokker-Planck equation describing the probability distribution function $\psi(t, \mathbf{x}, \mathbf{Q})$ of the dumbbell orientation $\mathbf{Q}$ on the microscopic level reads

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \psi = -\operatorname{div}_{\mathbf{Q}} \left((\nabla_{\mathbf{x}} \mathbf{u})^T \cdot \mathbf{Q} \psi - \frac{2}{\zeta} \mathbf{F}(\mathbf{Q}) \psi - \frac{2k\theta}{\zeta} \nabla_{\mathbf{Q}} \psi \right) \tag{9}$$

defined for $(t, \mathbf{x}, \mathbf{Q}) \in R_*^+ \times \Omega \times B$ and where ζ is the friction coefficient of the dumbbell beads, θ is the temperature, k is the Boltzmann constant, $\mathbf{F}$ is the spring force, Ω the physical fluid domain in $\mathbb{R}^d$ (with $d = 2$ or $d = 3$) and B the range for the elongation $\mathbf{Q}$, that is $B \subset \mathbb{R}^d$. The terms in (9) can be roughly explained as follows. The second term on the left-hand side of (9) stems from the fact that the polymers are convected by the macroscopic flow. The first term on the right-hand side of (9) stems from the fact that the polymers are stretched by this macroscopic flow and the last two terms account respectively for the inner force of the dumbbell due to the elongation, and the random collisions of the solvent particles with the polymers. However, for a more practical FENE model the spring force reads

$$\mathbf{F}(\mathbf{Q}) = \frac{H\mathbf{Q}}{1 - \frac{\|\mathbf{Q}\|^2}{Q_0^2}}$$

where H is the elastic constant and Q_0 is the maximum dumbbell extension and where $\|\cdot\|$ denotes the euclidien norm on $\mathbb{R}^d$, that is $\|\mathbf{Q}\|^2 = Q_1^2 + Q_2^2 + ... + \mathbf{Q}_d^2$ if $\mathbf{Q} = (Q_1, Q_2, ..., Q_d) \in \mathbb{R}^d$. In this case, $\mathbf{F}$ is defined for $\mathbf{Q} \in B(0, Q_0)$ and equation (9) stands for $B = B(0, Q_0)$.

Remark 2.1 *Other choices for the spring force can be used. The simplest one corresponds to the so-called Hookean dumbbells with* $\mathbf{F}(\mathbf{Q}) = H\mathbf{Q}$. *For the hookean dumbbells, the Fokker-Planck law* (9) *stands for* $B = \mathbb{R}^d$ *and leads to the Oldroyd-B fluid which is a macroscopic model. From a physical point of view, the Hookean potential is too simple and doesn't lead to a realistic description of the fluid since it permits to each polymer to have an infinite lenght. To obtain a more realistic macroscopic model than the Oldroyd-B model, there exists a closure, due to Peterlin, which consists in replacing the FENE spring force by the pre-average FENE-P approximation*

$$\mathbf{F}(\mathbf{Q}) = \frac{H\mathbf{Q}}{1 - \frac{\langle \mathbf{Q}^2 \rangle}{Q_0^2}} \qquad \text{where} \quad \langle \mathbf{Q}^2 \rangle = \int_{B(0,Q_0)} \mathbf{Q}^2 \psi(\mathbf{Q}) \, d\mathbf{Q}.$$

For the FENE-P model, it is possible to obtain a macroscopic model (see for instance [20]). Nevertheless, the Peterlin approximation can be very poor (see Keunings [21], Sizaire and al [38]), and much better closure approximations are available (Lielens and al. [24]). At any rate, closure-approximated dumbbell models (such *as FENE-P) are very useful in the development and evaluation of micro-macro methods, since the micro-macro results can be compared to those obtained with the continuum approach. It is important to notice that there does not exist an exact macroscopic constitutive equation for the FENE model, and thus the FENE model represents a truly multi-scale model.*

Introducing characteristic variables (denoted by a star $\star$, that is for instance replacing $\mathbf{x}$ by $L_\star \mathbf{x}$) we can write the Fokker-Planck equation (9) in the following non-dimensional form

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \psi = -\operatorname{div}_{\mathbf{Q}} \left((\nabla_{\mathbf{x}} \mathbf{u})^T \cdot \mathbf{Q}\psi - \frac{1}{2\mathcal{D}e} \mathbf{F}(\mathbf{Q})\psi - \frac{1}{2\mathcal{D}e} \nabla_{\mathbf{Q}} \psi \right) \tag{10}$$

defined for $(t, \mathbf{x}, \mathbf{Q}) \in R_*^+ \times \Omega \times B(0, \delta)$ and where $\mathcal{D}e$ and δ are two non-dimensional parameters given by

$$\mathcal{D}e = \frac{\zeta U_\star}{4 L_\star H} \qquad \text{and} \qquad \delta = \frac{Q_0}{Q_\star}. \tag{11}$$

The parameter $\mathcal{D}e$ is usually named the Deborah or the Weissenberg number. It is a comparison between two characteristic times : $T_\star^c = L_\star / U_\star$ which is the macroscopic convective time scale, and $T_\star^r = \zeta/(4H)$ which characterizes the mesoscopic relaxation time scale of the spring. The parameter δ is the comparison between two lengths: the maximal extensibility Q_0 of the dumbbells and the characteristic mean elongation $Q_\star$. According to H.C. Öttinger [32], the number δ roughly measures the number of monomer units represented by a bead and it is generally larger than 10.

Notice that in the non-dimensional model (10), the spring force writes $\mathbf{F}(\mathbf{Q}) = \dfrac{\mathbf{Q}}{1 - \|\mathbf{Q}\|^2/\delta^2}$.

Remark 2.2 *If $(\nabla_{\mathbf{x}} \mathbf{u})^T$ is replaced by its anti-symmetric part namely $\mathbf{W}(\mathbf{u}) = \frac{1}{2}(\nabla_{\mathbf{x}} \mathbf{u} - (\nabla_{\mathbf{x}} \mathbf{u})^T)$ in the Fokker-Plank equation (10) then we get the so-called co-rotational FENE model. The fact of putting $\mathbf{W}(\mathbf{u})$ instead of the whole $(\nabla_{\mathbf{x}} \mathbf{u})^T$ in (10) allows to get better estimates on ψ in a mathematical study. See for instance [27].*

As announced in the introduction, it is possible to write the Fokker-Planck equation (10) in a pleasant mathematical form. Precisely, we can find a function M such that $\operatorname{div}(\mathbf{F}\varphi + \nabla\varphi) = -\operatorname{div}(M\nabla(\frac{\varphi}{M}))$. First, remark that $M\nabla(\frac{\varphi}{M}) = \nabla\varphi - \nabla(\ln M)\varphi$. Hence, it is sufficient to find a function M such that $\nabla(\ln M) = \mathbf{F}$. Introducing

$$U(\mathbf{Q}) = -\frac{\delta^2}{2} \ln \left(1 - \frac{\|\mathbf{Q}\|^2}{\delta^2} \right),$$

we remark that $\nabla U = \mathbf{F}$ and then $\widetilde{M}(\mathbf{Q}) = \left(1 - \frac{\|\mathbf{Q}\|^2}{\delta^2}\right)^{\frac{\delta^2}{2}}$ gives the right behavior. In the literature, U is called the elastic spring potential and $\mathbf{F}$ is the spring force which derives from this potential. Since all functions on the form $\lambda\widetilde{M}$, $\lambda \in \mathbb{R}_+^*$ are appropriate, in the sequel we will prefer to work with the normalized Maxwellian

$$M(\mathbf{Q}) = \frac{1}{J} \left(1 - \frac{\|\mathbf{Q}\|^2}{\delta^2} \right)^{\delta^2/2} \quad \text{with} \quad J = \int_{B(0,\delta)} \left(1 - \frac{\|\mathbf{Q}\|^2}{\delta^2} \right)^{\delta^2/2} d\mathbf{Q}, \tag{12}$$

so that M satisfies $M \in \mathcal{C}^\infty(\overline{B(0,\delta)}, \mathbb{R})$, $0 < M \leq 1$ on $B(0,\delta)$, $M = 0$ on $\partial B(0,\delta)$ and $\int_{B(0,\delta)} M = 1$. Introducing this Maxwellian M, it is possible to re-write the Fokker-Planck equation (10) as follows:

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \psi = -\operatorname{div}_{\mathbf{Q}} \left((\nabla_{\mathbf{x}} \mathbf{u})^T \cdot \mathbf{Q}\psi \right) - \frac{1}{2\mathcal{D}e} \operatorname{div}_{\mathbf{Q}} \left(M \nabla_{\mathbf{Q}} \left(\frac{\psi}{M} \right) \right). \tag{13}$$

With these notations, we have the following result which will be used later to obtain estimates on the stress constraint $\boldsymbol{\sigma}$ from estimates on the density ψ.

Lemma 2.1 *Let $\mathbf{F}$ and M be respectively the spring force and the normalized Maxwellian introduced above. Assume that $\delta > \sqrt{2}$. Then we have the following estimate*

$$\int_{B(0,\delta)} M(\mathbf{Q}) |\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}|^2 d\mathbf{Q} < +\infty$$

where $|\cdot|$ corresponds to the following norm on the 2-tensors : $|\mathbf{A}| = \sup_{i,j} |A_{i,j}|$.

Proof: According to the definition of the norm $|\cdot|$, it suffices to show that each component

$$\mathcal{M}_{i,j} = \int_{B(0,\delta)} \widetilde{M}(\mathbf{Q})\,(F(\mathbf{Q})_i Q_j)^2\, dQ_1...dQ_d,$$

for $(i,j) \in \{1,...,d\}^2$, is finite (for sake of simplicity, we present the demonstration by using the non-normalized form $\widetilde{M}$ of the Maxwellian M, knowing that this result clearly implies the one for the normalized maxwellian M). By definition of the Maxwellian $\widetilde{M}$ and of the spring force $\mathbf{F}$, we get

$$\mathcal{M}_{i,j} = \int_{B(0,\delta)} Q_i^2 Q_j^2 \left(1 - \frac{\|\mathbf{Q}\|^2}{\delta^2}\right)^{\frac{\delta^2-4}{2}} dQ_1...dQ_d.$$

Using the polar coordinate, that is the change of coordinate

$$\Phi : (r;\theta_1,...,\theta_{d-1}) \in \mathbb{R}_+^* \times S^{d-1} \longmapsto \begin{pmatrix} r\sin\theta_1 \\ r\cos\theta_1\sin\theta_2 \\ \vdots \\ r\cos\theta_1 ... \cos\theta_{d-1} \end{pmatrix} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$$

whose the Jacobian is of the form $\mathrm{Jac}(\Phi)(r,\theta) = r^{d-1}\widehat{\Phi}(\theta)$ (the function $\widehat{\Phi}$ being a continuous function on the sphere S^{d-1}), we get

$$\mathcal{M}_{i,j} = \left(\int_{S^{d-1}} \widehat{\Phi}(\theta) d\theta\right)\left(\int_0^{\delta} r^{3+d}\left(1 - \frac{r^2}{\delta^2}\right)^{\frac{\delta^2-4}{2}} dr\right) = C\int_0^{\delta^2} s^{1+d/2}\left(1 - \frac{s}{\delta^2}\right)^{\frac{\delta^2-4}{2}} ds.$$

The result follows from the assumption $\delta > \sqrt{2}$ since $\int_0^1 s^\alpha ds$ converges as soon as $\alpha > -1$. $\square$

2.2 Stress tensor for the FENE model

The total Cauchy stress tensor $\boldsymbol{\sigma}$ (without taking account of the pressure effect already introduced in the momentum equation *via* the term ∇p) can be decomposed into the sum

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_S + \boldsymbol{\sigma}_P$$

where $\boldsymbol{\sigma}_S$ denotes the solvent contribution and $\boldsymbol{\sigma}_P$ the sum of the spring tension contribution and the bead motion contribution. The expressions for all these contributions in the homogeneous flow case can be found in the book of Bird and al. [7]. We use their extensions to the non-homogeneous flow case developed by Biller and Petruccione [6, 33], see also [19]. For the solvent contribution we use the classical Newtonian stress:

$$\boldsymbol{\sigma}_S = \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

where η correspond to the solvent viscosity of the fluid. The contribution of polymer to the constraint written in dimensional form makes the absolute temperature of the flow θ and the Boltzmann constant k appear. It writes:

$$\boldsymbol{\sigma}_P = \langle \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\rangle_{\dim} - k\theta\langle \mathbf{Id}\rangle_{\dim} \tag{14}$$

where the average $\langle\cdot\rangle_{\dim}$ over $\mathbf{Q}$-space for a quantity f is defined by

$$\langle f\rangle_{\dim} = \int_{B(0,Q_0)} f(\mathbf{Q})\psi(\mathbf{Q})d\mathbf{Q}.$$

To write the contribution $\boldsymbol{\sigma}_P$ of polymer to the constraint in a non-dimensional form we use the non-dimensional number δ (see its definition given in equation (11)) and introduce an additional non-dimensional number $\lambda = \frac{k\theta Q_\star L_\star}{\eta U_\star}$. Hence, in a non-dimensional form, the relation (14) reads

$$\boldsymbol{\sigma}_P = \lambda\Big(\langle \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\rangle - \rho\mathbf{Id}\Big). \tag{15}$$

In this formulation, the average $\langle \cdot \rangle$ for a quantity f is defined by

$$\langle f \rangle = \int_{B(0,\delta)} f(\mathbf{Q})\psi(\mathbf{Q})d\mathbf{Q}$$

and the notation $\langle 1 \rangle = \displaystyle\int_{B(0,\delta)} \psi(\mathbf{Q})d\mathbf{Q}$ corresponds to the density of the polymer chains and is denoted by ρ.

2.3 Explicit solution for the Fokker-Planck equation

Equilibrium solution - There exists a very simple case for which we know the exact explicit solution to the Fokker-Planck equation (10). It is a naturally stationary solution corresponding to the equilibrium case (that is with $\mathbf{u}_{\text{eq}} = \mathbf{0}$):

$$\psi_{\text{eq}}(\mathbf{Q}) = \rho M(\mathbf{Q}),$$

where the constant ρ corresponds to the density of the polymer chains, that is $\rho = \int_{B(0,\delta)} \psi_{\text{eq}}$.

Steady state and co-rotational case - In the co-rotational case (that is when the quantity $\nabla_{\mathbf{x}}\mathbf{u}$ is replaced by the skew-symmetric tensor $\mathbf{W} = \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} - (\nabla_{\mathbf{x}}\mathbf{u})^T\right)$ in equation (10), see also Remark 2.2) the stationary solution is explicitly given by $\psi_{\text{eq}}(\mathbf{Q}) = \rho M(\mathbf{Q})$. Indeed, in this case we have (using the Einstein summation convention)

$$\begin{aligned}
\operatorname{div}_{\mathbf{Q}}(\mathbf{W}(\mathbf{x}) \cdot \mathbf{Q}\psi_{\text{eq}}(\mathbf{Q})) &= \rho\partial_{Q_i}\Big(W_{ij}(\mathbf{x})Q_j M(\mathbf{Q})\Big) \\
&= \rho W_{ij}(\mathbf{x})\delta_{ij}M(\mathbf{Q}) + W_{ij}(\mathbf{x})Q_j\partial_{Q_i}(M(\mathbf{Q})) \\
&= \rho W_{ii}(\mathbf{x})M(\mathbf{Q}) + W_{ij}(\mathbf{x})Q_jQ_iN(\mathbf{Q})
\end{aligned}$$

where we remark that we can write an equality on the form $\partial_{Q_i}(M(\mathbf{Q})) = Q_iN(\mathbf{Q})$. Since the tensor W is skew-symmetric and the tensor $\mathbf{Q} \otimes \mathbf{Q}$ is symmetric, we easily deduce that

$$\operatorname{div}_{\mathbf{Q}}(\mathbf{W}(\mathbf{x}) \cdot \mathbf{Q}\psi_{\text{eq}}(\mathbf{Q})) = 0.$$

Homogeneous flows - More generally, $\mathbf{u}$ is a so-called homogeneous velocity field if there exists a tensor $\boldsymbol{\tau}(t)$, a point $\mathbf{x}_0 \in \Omega$ and a constant vector $\mathbf{u}_0 \in \mathbb{R}^d$ such that $\mathbf{u}(t, \mathbf{x}) = \boldsymbol{\tau}(t) \cdot (\mathbf{x} - \mathbf{x}_0) + \mathbf{u}_0)$. In these cases, it is natural to consider a solution ψ to (10) which does not depend on macroscopic space, that is which does not depend on the variable $\mathbf{x}$. Classical examples of homogeneous flows (see, for example, Chapter 3 in [7] or [26, p. 9-11]) are shear flows and elongational flows.

Steady state extentional flows - Two important special homogeneous flows are planar extensional flow and uniaxial extensional flows with the velocity gradient respectively given by

$$\boldsymbol{\tau} = \dot{\varepsilon}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau} = \dot{\varepsilon}\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In both cases, $\dot{\varepsilon}$ is called the extensional rate. The Fokker-Planck equation (10) has a steady-state analytical solution for both types of extensional flows (and more generally for any symmetric matrix $\boldsymbol{\tau}$). This solution is given by formula (13.2-14) in [7] and has the form

$$\psi(\mathbf{Q}) = M(\mathbf{Q})\exp\left(\mathcal{D}e\,\boldsymbol{\tau} : \mathbf{Q} \otimes \mathbf{Q}\right).$$

Steady state shear flows - It corresponds to a flow where the velocity gradient is in the following form

$$\boldsymbol{\tau} = \dot{\gamma}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

the coefficient $\dot{\gamma}$ is called the shear rate. The stationary solution of the Fokker-Planck equation (10) cannot be found analytically, but it is relatively easy to construct an approximation in the limit of small $\dot{\gamma}$ (see

Equation 13.5-15, p. 79 of [7]). To see this, we represent ψ as $\psi = \psi_{\text{eq}}(1 + \mathcal{D}e\,\dot\gamma\,\psi_1 + \mathcal{D}e^2\,\dot\gamma^2\,\psi_2 + ...)$, substitute it into (10) to obtain the equation for each term ψ_k. We get

$$\begin{aligned}\psi(\mathbf{x},\mathbf{Q}) = \rho\, M(\mathbf{Q})\Big(1+\frac{\mathcal{D}e}{2}\mathbf{D}(\mathbf{x}) : \mathbf{Q}\otimes\mathbf{Q} + \frac{\mathcal{D}e^2}{4}\Big(\frac{1}{2}(\mathbf{D}(\mathbf{x}) : \mathbf{Q}\otimes\mathbf{Q})^2 - \frac{1}{15}\langle\|\mathbf{Q}\|^4\rangle_{\text{eq}}\mathbf{D}(\mathbf{x}) : \mathbf{D}(\mathbf{x})\\ + \frac{4\delta^2}{2\delta^2+7}(1 - \frac{\|\mathbf{Q}\|^2}{2\delta^2})(\mathbf{D}(\mathbf{x})\cdot\mathbf{W}(\mathbf{x})) : \mathbf{Q}\otimes\mathbf{Q}\Big) + \mathcal{O}(\mathcal{D}e^3)\Big),\end{aligned} \tag{16}$$

where the notation $\langle\,\cdot\,\rangle_{\text{eq}}$ corresponds to $\int_B \cdot\,\rho M(\mathbf{Q})\,d\mathbf{Q}$, and $\mathbf{D}(\mathbf{x})$ and $\mathbf{W}(\mathbf{x})$ are respectively the symmetric and skew-symmetric part of the velocity gradient $\nabla_{\mathbf{x}}\mathbf{u} = \boldsymbol{\tau}$.
Note that this kind of development was obtained again by P. Degond, M. Lemou and M. Picasso [13]. They recover the calculations of [7] using a different method: the probabilist density ψ is expand in powers of the Weissenberg number $\mathcal{D}e$ using a Chapman-Enskog expansion technique.

Remark 2.3 *In part 4, we will study the Fokker-Planck equation* (10) *in the stationary stationary case without the convective term* $\mathbf{u}\cdot\nabla_{\mathbf{x}}\psi$ *but for a more general velocity than in [7], that is for a non-homogeneous flow and for a non small velocity. We will not find an explicit formula for* ψ *but the analysis will make it possible to prove that the approximation* (16) *is justified in non-homogeneous cases.*

3 Functional spaces and fundamental lemmas

3.1 Functional spaces

From the peculiar form of the Fokker-Planck equation (13), the adapted functional spaces use Sobolev weight spaces on the ball $B = B(0,\delta)$. More precisely, we introduce

$$L^2(B;M) := \{g \in L^1_{loc}(B)\ ;\ \int_B M|g|^2 < +\infty\},$$
$$H^1(B;M) := \{g \in L^1_{loc}(B)\ ;\ \int_B M|g|^2 + M|\nabla g|^2 < +\infty\}.$$

These two spaces are Hilbert spaces (see for instance H. Triebel [41, Th. 3.2.2a]) and are endowed with their usual norms respectively denoted $\|\cdot\|_{L^2(B;M)}$ and $\|\cdot\|_{H^1(B;M)}$. In the same way, we introduce

$$L^2_M := M.L^2(B;M) = \{\varphi \in L^1_{loc}(B)\ ;\ \int_B M\left|\frac{\varphi}{M}\right|^2 < +\infty\},$$
$$H^1_M := M.H^1(B;M) = \{\varphi \in L^1_{loc}(B)\ ;\ \int_B M\left|\frac{\varphi}{M}\right|^2 + M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2 < +\infty\}.$$

Their natural norms are denoted $\|\cdot\|_0$ and $\|\cdot\|_1$. By definition, they satisfy

$$\|\psi\|_0 = \left\|\frac{\psi}{M}\right\|_{L^2(B;M)} \qquad \text{and} \qquad \|\psi\|_1 = \left\|\frac{\psi}{M}\right\|_{H^1(B;M)}.$$

Remark 3.1 *Since* $0 < M \le 1$ *on* B *we can observe that* $L^2_M \subset L^2(B)$ *and* $H^1_M \subset H^1(B)$ *where the spaces* $L^2(B)$ *and* $H^1(B)$ *are the classical Sobolev spaces on the set* B.

3.2 Linear operator

One more important ingredient in our study is the following linear operator

$$\mathcal{L}\psi = -\operatorname{div}\Big(M\frac{\psi}{M}\Big)$$

on the space L^2_M and with domain, see [27, Remark 3.8, p. 9] and notice that we have needed the assumption $\delta \ge \sqrt{2}$, given by

$$D(\mathcal{L}) = \{\psi \in H^1_M\ ;\ \int_B \frac{1}{M}\Big|\operatorname{div}\Big(M\frac{\psi}{M}\Big)\Big|^2 < +\infty\}.$$

We also find in [27, Proposition 3.6, p. 8] the following result and its proof which will be used to introduce the Galerkin approximation method later.

Lemma 3.1 *The operator $\mathcal{L}$ is self-adjoint and positive. Moreover, it has a discrete spectrum formed by a sequence $(\ell_n)_{n\in\mathbb{N}}$ such that ℓ_n goes to $+\infty$ when n goes to $+\infty$.*

About uniqueness results for a linear operator, it is known that the eigenvalue 0, that is the kernel of the operator $\mathcal{L}$, is particularly important.

Lemma 3.2 *The kernel of the operator $\mathcal{L}$ is the set $\{\lambda M, \lambda \in \mathbb{R}\}$.*

Proof: This lemma is an immediate consequence of the following formulation of the operator $\mathcal{L}$:

$$\langle \mathcal{L}\psi, \varphi\rangle_{L^2_M} = \int_B M\nabla\left(\frac{\psi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right) \tag{17}$$

where $\langle\cdot,\cdot\rangle_{L^2_M}$ corresponds to the scalar product subordinated to the norm $\|\cdot\|_0$ on L^2_M. Indeed, let ψ such that $\mathcal{L}\psi = 0$. We easily get $\langle\mathcal{L}\psi,\psi\rangle_{L^2_M} = 0$ and the formulation (17) yields $\nabla\left(\frac{\psi}{M}\right) = 0$. Thus, thanks to the connexity of B, we deduce that there exists $\lambda\in\mathbb{R}$ such that $\psi = \lambda M$. $\square$

It is natural to introduce the following normalized subspace

$$H^1_{M,0} = \{\psi \in H^1_M \ ; \ \int_B \psi = 0\},$$

so that, since $\int_B M = 1$, the kernel of $\mathcal{L}\big|_{H^1_{M,0}}$ is the null space $\{0\}$. In the sequel, the space $H^1_{M,0}$ will by equipped with the norm

$$\|\psi\|_{1,0} = \sqrt{\int_B M\left|\nabla\left(\frac{\psi}{M}\right)\right|^2}.$$

Knowing the kernel of the operator $\mathcal{L}$, we deduce that $\|\cdot\|_{1,0}$ is really a norm on the space $H^1_{M,0}$. It is also a semi-norm on the space H^1_M and notation $\|\psi\|_{1,0}$ will be sometimes used also for functions ψ in H^1_M.
To build functions in $H^1_{M,0}$ we use the following lemma which will be important to obtain a lot of test functions in the weak formulation later.

Lemma 3.3 *Let $\psi\in H^1_M$ and $\xi:\mathbb{R}\to\mathbb{R}$ be a continuous application, piecewise-$\mathcal{C}^1$ such that ξ' is bounded on $\mathbb{R}$. Then we have*

$$\varphi := M\xi\left(\frac{\psi}{M}\right) - M\int_B M\xi\left(\frac{\psi}{M}\right) \in H^1_{M,0}$$

and $\nabla\left(\frac{\varphi}{M}\right) = \xi'\left(\frac{\psi}{M}\right)\nabla\left(\frac{\psi}{M}\right)$. Moreover, we have $\|\varphi\|_{1,0}\le\|\xi'\|_\infty\|\psi\|_{1,0}$.

Main ideas of the proof: This result is inspired by the main steps of the proof of the Stampacchia lemma which affirms that if $g\in H^1(B)$ and $\xi:\mathbb{R}\to\mathbb{R}$ is continuous, piecewise-$\mathcal{C}^1$, such that ξ' is bounded on $\mathbb{R}$ then we have $\xi(g)\in H^1(B)$ and $\nabla\xi(g) = \xi'(g)\nabla g$. First we prove the result for a regular function $\xi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ approaching $\psi\in H^1_M$ by a sequence $\psi_n\in\mathcal{C}^\infty(\overline{B},\mathbb{R})$ (indeed the space $C^\infty(\overline{B},\mathbb{R})$ is dense in $H^1(B,M)$, see for instance [1, Proof of lemma 3.1] or [41, Theorem 3.2.2c]). In the less regular cases for the function ξ the only difference comes from to the fact that the function ξ has discontinuity points; the set Z of discontinuity points being a subset of $\mathbb{R}$ with null Lebesgue measure. We use the following result (see [15]):

$$\nabla\left(\frac{\psi}{M}\right) = 0 \quad \text{almost everywhere on} \quad \left\{\mathbf{Q}\in B\ ;\ \frac{\psi(\mathbf{Q})}{M(\mathbf{Q})}\in Z\right\}.$$

These arguments prove that $M\xi\left(\frac{\varphi}{M}\right)\in H^1_M$ and that $\nabla\xi\left(\frac{\varphi}{M}\right) = \xi'\left(\frac{\psi}{M}\right)\nabla\left(\frac{\psi}{M}\right)$. The fact that φ is null average is then immediate since $\int_B M = 1$. The inequality on the norm comes from the following estimate

$$\|\varphi\|_{1,0} = \sqrt{\int_B M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2} = \sqrt{\int_B M\xi'\left(\frac{\psi}{M}\right)^2\left|\nabla\left(\frac{\psi}{M}\right)\right|^2} \le \|\xi'\|_\infty\|\psi\|_{1,0}.$$

$\square$

3.3 Fundamental lemmas

The next lemma is a generalized Poincaré inequality adapted to the weighted spaces introduced before. Its proof can be found in H.J. Brascamp [9] (see also Proposition 2.1 in [13]) and use the fact that the function U is strictly uniformly convex. More precisely the Hessian matrix of U satisfies

$$\text{Hess } U(\mathbf{Q}) = \frac{1}{1-\frac{\|\mathbf{Q}\|^2}{\delta^2}}\left(\mathbf{Id} + \frac{2\mathbf{Q}\otimes\mathbf{Q}}{\delta^2-\|\mathbf{Q}\|^2}\right)$$

which implies that for all $\mathbf{Q} \in B$, i. e. such that $\|\mathbf{Q}\| \leq \delta$, and for all $\mathbf{x} \in \mathbb{R}^d$ we have

$$\mathbf{x}^T \cdot \text{Hess } U(\mathbf{Q}) \cdot \mathbf{x} \; \geq \frac{1}{1-\frac{\|\mathbf{Q}\|^2}{\delta^2}}\mathbf{x}^T \cdot \mathbf{x} \; \geq \mathbf{x}^T \cdot \mathbf{x}.$$

Using a result of H.J. Brascamp [9], we thus have

Lemma 3.4 *For all $\varphi \in H^1_M$ we get the following Poincaré-type inequality*

$$\int_B M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2 + \left(\int_B \varphi\right)^2 \geq \|\varphi\|_0^2.$$

For the free-average functions (that is for $\psi \in H^1_{M,0}$), this lemma 3.4 show that the two norms $\|\cdot\|_1$ and $\|\cdot\|_{1,0}$ are equivalents. This equivalence will be usually useful in the rest of the paper.

Lemma 3.5 *The injection $H^1_M \subset L^2_M$ is compact.*

Proof: Consider a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ bounded in H^1_M and show that a convergent sub-sequence can be extracted. By definition of H^1_M, for all $n \in \mathbb{N}$, there exists $g_n \in H^1(B; M)$ such that $\varphi_n = Mg_n$. The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ being bounded in H^1_M, the sequence $\{g_n\}_{n\in\mathbb{N}}$ is bounded in L^2_M. Since $M > 0$ on B, $M = 0$ on ∂B and $dM \neq 0$ out of ∂B, we can use the result of G. Métivier [29, Proposition 3.1 p. 221] affirming that the weight Sobolev space injection $H^1(B; M) \subset L^2(B; M)$. Thus, we can extract from the sequence $\{g_n\}_{n\in\mathbb{N}}$ a sub-sequence, still noted $\{g_n\}_{n\in\mathbb{N}}$ and such that

$$g_n \rightharpoonup g \quad \text{in } H^1(B; M) \qquad \text{and} \qquad g_n \to g \quad \text{in } L^2(B; M).$$

By definition of the spaces H^1_M and L^2_M we conclude that

$$\varphi_n \rightharpoonup Mg \quad \text{in } H^1_M \qquad \text{and} \qquad \varphi_n \to Mg \quad \text{in } L^2_M$$

which proves that the injection $H^1_M \hookrightarrow L^2_M$ is compact. $\square$

In the same way, we use this classical next lemma which proves that functions in H^1_M are in L^p_M where the weighted-space L^p_M is defined by

$$L^p_M = \left\{\varphi \in L^1_{loc}(B) \; ; \; \left(\int_B M\left|\frac{\varphi}{M}\right|^p\right)^{1/p} < +\infty\right\}$$

and endowed with its usual norm. More exactly, we have

Lemma 3.6 *The injection $H^1_M \subset L^p_M$ is continuous for all $2 \leq p \leq \dfrac{2d}{d-2}$.*

The last lemma of this part concerns the traces on ∂B for function $\psi \in H^1_M$. In fact, we can observe that we never have introduce boundaries conditions associated with the Fokker-Planck equation (10) whereas it make appear the second order operator $\mathcal{L}$. This comes from the following result whose the proof is given in [27, Remarks 3.7 and 3.8, p. 9].

Lemma 3.7 *If $\delta > \sqrt{2}$ then for $\psi \in H^1_M$ the trace of ψ on ∂B exists and is equal to 0.*

About the boundaries conditions, a recent paper of C. Liu and H. Liu [25] shows that for the Fokker-Planck equation, any pre-assigned distribution on boundary will become redundant once $\delta \geq \sqrt{2}$. Moreover if the probability density function ψ is regular enough for its trace to be defined on ∂B (for instance if $\psi \in H^1_M$) then the trace is necessarily zero when $\delta > \sqrt{2}$. So the appropriate function space for weak solution may be chosen as a subspace of the usual Hilbert space, restricted with a proper weight to take care of the boundary singularity. From the lemma 3.7 all subspace of the Hilbert space H^1_M is appropriate for the Fokker-Planck equation.

Finally, we denote H^{-1}_M the topological dual of $H^1_{M,0}$, that is the set of continuous linear forms on $H^1_{M,0}$. Each application $\chi \in H^{-1}_M$ will be defined by $\chi : \varphi \in H^1_{M,0} \mapsto \langle \chi, \varphi \rangle \in \mathbb{R}$. By its continuity, for each $\chi \in H^{-1}_M$ there exists $C \in \mathbb{R}$ such that

$$\forall \varphi \in H^1_{M,0} \quad |\langle \chi, \varphi \rangle| \leq C \|\varphi\|_{1,0}.$$

As it is usual, the smallest of these constants C is denoted $\|\chi\|_{-1}$: it is the norm of χ on H^{-1}_M.

4 Stationary solution

4.1 Main results

Let $\mathbf{u} \in W^{1,\infty}(\Omega)$ be a velocity vector field on a bounded open subset Ω of $\mathbb{R}^d$, $\rho \in L^\infty(\Omega)$ be a density polymer chains scalar field on Ω and $\mathbf{F}$ the spring elastic force of the FENE polymer model (that is $\mathbf{F}(\mathbf{Q}) = \frac{1}{1-\|\mathbf{Q}\|^2/\delta^2}$ for $\mathbf{Q} \in B$ where B is the ball $B(0,\delta) \subset \mathbb{R}^d$). We show in this part that there exists an unique solution ψ depending on both $\mathbf{x}$ the macroscopic variable and $\mathbf{Q}$ the microscopic one, to the following Fokker-Planck equation

$$\operatorname{div}_{\mathbf{Q}} \left((\nabla_{\mathbf{x}} \mathbf{u})^T \cdot \mathbf{Q}\psi - \frac{1}{2\mathcal{D}e} \mathbf{F}(\mathbf{Q})\psi - \frac{1}{2\mathcal{D}e} \nabla_{\mathbf{Q}} \psi \right) = 0 \tag{18}$$

on $\Omega \times B$ and satisfying $\int_B \psi(\mathbf{x}, \mathbf{Q}) d\mathbf{Q} = \rho(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. It is clear that the variable $\mathbf{x}$ can be viewed as a parameter and in this part and we can consider that there is only one variable $\mathbf{Q}$. All the operators used in this part will refer to this variable, that is $\nabla_{\mathbf{Q}} = \nabla$, $\operatorname{div}_{\mathbf{Q}} = \operatorname{div}$,...
Using the part 2, equation (18) can be rewritten as

$$-\frac{1}{2\mathcal{D}e} \operatorname{div} \left(M \nabla \left(\frac{\psi}{M} \right) \right) + \operatorname{div} (\psi \boldsymbol{\kappa}) = 0 \tag{19}$$

on B where M is a Maxwellian on B defined by the relation (12) and $\boldsymbol{\kappa}$ corresponds to the velocity influence, that is in a classical framework $\boldsymbol{\kappa} = (\nabla_{\mathbf{x}} \mathbf{u})^T \cdot \mathbf{Q}$. Exchanging ψ into $\psi - \rho M$, since $\int_B M = 1$, we will be interested here in the equivalent equation

$$-\frac{1}{2\mathcal{D}e} \operatorname{div} \left(M \nabla \left(\frac{\psi}{M} \right) \right) + \operatorname{div} (\psi \boldsymbol{\kappa}) = f$$

on B with $\int_B \psi = 0$ and $f = \rho \operatorname{div}(M\boldsymbol{\kappa})$. The weak formulation of this equation writes: find $\psi \in H^1_{M,0}$ such that for all $\varphi \in H^1_{M,0}$

$$\frac{1}{2\mathcal{D}e} \int_B M \nabla \left(\frac{\psi}{M} \right) \cdot \nabla \left(\frac{\varphi}{M} \right) - \int_B \psi \boldsymbol{\kappa} \cdot \nabla \left(\frac{\varphi}{M} \right) = \langle f, \varphi \rangle \tag{20}$$

where $\langle \cdot, \cdot \rangle$ denote the duality brackets between H^{-1}_M and $H^1_{M,0}$. We prove in this part the following theorem.

Theorem 4.1 *Let $B = B(0,\delta)$ with $\delta > \sqrt{2}$, $\boldsymbol{\kappa} \in L^\infty(B, \mathbb{R}^d)$ and $M \in \mathcal{C}^\infty(\overline{B}, \mathbb{R})$ be a normalized Maxwellian*[2]. *For all $f \in H^{-1}_M$ the problem (20) admits an unique solution $\psi \in H^1_{M,0}$.*

Hence, taking $f = \rho \operatorname{div}(M\boldsymbol{\kappa})$ and $\psi \in H^1_{M,0}$ the solution to equation (20) for this term source, we deduce that $\psi + \rho M$ is the unique weak solution to equation (19). Thus for $\rho \in \mathcal{C}(\Omega, \mathbb{R})$ the theorem 4.1 implies the existence of a weak solution $\psi \in \mathcal{C}(\Omega, H^1_M)$ to equation (18) such that for all $\mathbf{x} \in \Omega$ we have $\int_B \psi(\mathbf{x}, \mathbf{Q})\, d\mathbf{Q} = \rho(\mathbf{x})$.

[2] That is to say that the function M satisfies $0 < M \leq 1$ on B, $M = 0$ on ∂B and $\int_B M = 1$

Remark 4.1

- As specified before, the assumption $\delta > \sqrt{2}$ is not constraining from the physical point of view since δ is generally larger than 10.

- Recall that in the co-rotational case, that is when $\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q}) = \mathbf{W}(\mathbf{x})\cdot\mathbf{Q}$ with $\mathbf{W} = \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} - (\nabla_{\mathbf{x}}\mathbf{u})^T\right)$, the solution to (19) *is explicitly given by $\psi_{eq} = \rho M$, see section 2.3.*

4.2 Existence proof in theorem 4.1

Principle for the existence proof of theorem 4.1 - Using the equivalence between the norm $\|\cdot\|_1$ and $\|\cdot\|_{1,0}$ on the space $H^1_{M,0}$, see lemma 3.4, the operator $\varphi \mapsto -\operatorname{div}\left(M\nabla\left(\frac{\varphi}{M}\right)\right)$ is coercitiv on $H^1_{M,0}$, thus we can (see for instance the Lax-Milgram theorem) proof that there exists a weak solution (that is belonging to $H^1_{M,0}$) to equations like

$$-\frac{1}{2\mathcal{D}e}\operatorname{div}\left(M\nabla\left(\frac{\psi}{M}\right)\right) = f$$

as soon as the source term f belongs in H^{-1}_M. Moreover in this case we have $\|\psi\|_1 \le C\|f\|_{H^{-1}_M}$ where C is constant only depending on the domain B and on the Deborah number $\mathcal{D}e$.
Because of the non-coercivity of the operator $\varphi \mapsto -\operatorname{div}\left(M\nabla\left(\frac{\varphi}{M}\right)\right) + \operatorname{div}(\varphi\boldsymbol{\kappa})$, we start by studying an approximate problem. For each $n \in \mathbb{N}$, let us consider the application $T_n : r \in \mathbb{R} \mapsto \max(\min(r,n),-n) \in \mathbb{R}$ (see also figure 2 below) and let us denote by F_n the following application: $F_n : \widetilde{\psi} \in L^2_M \mapsto \psi \in H^1_{M,0} \subset L^2_M$ where ψ is the weak solution of

$$-\frac{1}{2\mathcal{D}e}\operatorname{div}\left(M\nabla\left(\frac{\psi}{M}\right)\right) = f - \operatorname{div}\left(MT_n\left(\frac{\widetilde{\psi}}{M}\right)\boldsymbol{\kappa}\right). \tag{21}$$

We will note that for $\widetilde{\psi} \in L^2_M$ we have $MT_n\left(\frac{\widetilde{\psi}}{M}\right) \in L^2_M$ and since $\boldsymbol{\kappa} \in L^\infty(\Omega)$ we get $\operatorname{div}\left(MT_n\left(\frac{\widetilde{\psi}}{M}\right)\boldsymbol{\kappa}\right) \in H^{-1}_M$. The function F_n is then well defined.

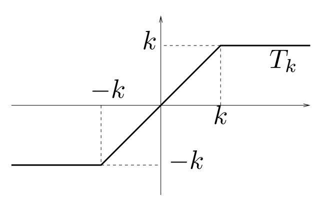


Figure 2: The function T_k, $k \in \mathbb{N}^*$.

Let us prove that F_n is a compact application by showing that its image $F_n(L^2_M)$ is bounded in L^2_M. Let us consider $\psi = F_n(\widetilde{\psi}) \in F_n(L^2_M)$. Taking ψ as test function in the weak formulation of the equation (21) we get

$$\frac{1}{2\mathcal{D}e}\int_B M\left|\nabla\left(\frac{\psi}{M}\right)\right|^2 = \langle f,\psi\rangle + \int_B MT_n\left(\frac{\widetilde{\psi}}{M}\right)\boldsymbol{\kappa}\cdot\nabla\left(\frac{\psi}{M}\right).$$

In other words, by using the duality definition and the Cauchy-Schwarz inequality, we have

$$\|\psi\|^2_{1,0} \le C\|\psi\|_{1,0} + 2\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty\sqrt{\int_B M\left|T_n\left(\frac{\widetilde{\psi}}{M}\right)\right|^2}\sqrt{\int_B M\left|\nabla\left(\frac{\psi}{M}\right)\right|^2}.$$

Using the fact successively that for all $r \in \mathbb{R}$ we have $|T_n(r)| \le n$ and that $\int_B M = 1$ we deduce that

$$\|\psi\|^2_{1,0} \le C\|\psi\|_{1,0} + 2n\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty\sqrt{\int_B M}\|\psi\|_{1,0} = C\|\psi\|_{1,0} + 2n\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty\|\psi\|_{1,0}.$$

We deduce that

$$\|F_n(\widetilde{\psi})\|_0 = \|\psi\|_0 \le \|\psi\|_{1,0} \le C + 2n\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty.$$

Thus, the image of L^2_M by the application F_n is contained in the ball of L^2_M of radius $C+2n\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty$. Applying the Schauder fixed point theorem, we conclude that the application F_n admits a fixed point, denoted by ψ_n, in L^2_M. This fixed point is consequently solution of

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi_n}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right) - \int_B MT_n\left(\frac{\psi_n}{M}\right)\boldsymbol{\kappa}\cdot\nabla\left(\frac{\varphi}{M}\right) = \langle f,\varphi\rangle \qquad \forall\varphi\in H^1_{M,0}. \tag{22}$$

The continuation of the proof consists in obtaining estimates on these functions ψ_n in order to be able to pass at the limit when n tends to $+\infty$.

Estimate of $\boldsymbol{M\ln\left(1+\left|\frac{\psi_n}{M}\right|\right)}$ in $\boldsymbol{H^1_{M,0}}$-norm - Let ξ be the application from $\mathbb{R}$ to $\mathbb{R}$ defined by $\xi(r) = \displaystyle\int_0^r \frac{ds}{(1+|s|)^2}$. This application is continuous, piecewise-$\mathcal{C}^1$ and with bounded derivative. According to lemma 3.3 we can choose $\varphi = M\xi\left(\frac{\psi_n}{M}\right) - M\displaystyle\int_B M\xi\left(\frac{\psi_n}{M}\right)$ as test function in formulation (22). The first of the three terms obtained is treated in the following way

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi_n}{M}\right)\cdot\nabla\Big(\xi\left(\frac{\psi_n}{M}\right)\Big) = \frac{1}{2\mathcal{D}e}\int_B M\frac{\left|\nabla\left(\frac{\psi_n}{M}\right)\right|^2}{\left(1+\left|\frac{\psi_n}{M}\right|\right)^2} = \frac{1}{2\mathcal{D}e}\Big\|M\ln\Big(1+\left|\frac{\psi_n}{M}\right|\Big)\Big\|^2_{1,0}. \tag{23}$$

For the second term we get

$$\left|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(\xi\left(\frac{\psi_n}{M}\right)\Big)\right| = \left|\int_B \frac{MT_n\left(\frac{\psi_n}{M}\right)}{1+\left|\frac{\psi_n}{M}\right|}\boldsymbol{\kappa}\cdot\frac{\nabla\left(\frac{\psi_n}{M}\right)}{1+\left|\frac{\psi_n}{M}\right|}\right| \le \|\boldsymbol{\kappa}\|_\infty\int_B\left|\frac{T_n\left(\frac{\psi_n}{M}\right)}{1+\left|\frac{\psi_n}{M}\right|}\right| M\left|\nabla(\ln\Big(1+\left|\frac{\psi_n}{M}\right|\Big))\right|.$$

Using the fact that for all $r\in\mathbb{R}$ we have $|T_n(r)|\le|r|$, we deduce that[3]

$$\left|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(\xi\left(\frac{\psi_n}{M}\right)\Big)\right| \le C\Big\|M\ln\Big(1+\left|\frac{\psi_n}{M}\right|\Big)\Big\|_{1,0}. \tag{24}$$

For the last term since $f\in H^{-1}_M$ we deduce

$$|\langle f,\varphi\rangle| \le C\|\varphi\|_{1,0} = C\sqrt{\int_B M\xi'\left(\frac{\psi_n}{M}\right)\nabla\left(\frac{\psi_n}{M}\right)} = C\sqrt{\int_B M\frac{\left|\nabla\left(\frac{\psi_n}{M}\right)\right|^2}{\left(1+\left|\frac{\psi_n}{M}\right|\right)^2}} = C\Big\|M\ln\Big(1+\left|\frac{\psi_n}{M}\right|\Big)\Big\|_{1,0}. \tag{25}$$

The three estimate (23), (24) and (25) allow us to obtain the existence of a constant $C>0$ such that for all $n\in\mathbb{N}$ we have

$$\Big\|M\ln\Big(1+\left|\frac{\psi_n}{M}\right|\Big)\Big\|_{1,0} \le C. \tag{26}$$

Estimate of $\boldsymbol{\mu(\{\mathbf{Q}\in B\ ;\ |\psi_n(\mathbf{Q})|\ge kM(\mathbf{Q})\})}$ - In this paragraph, we control the size of the set where ψ_n take large values, that is the set $\mathcal{E}_k = \{\mathbf{Q}\in B\ ;\ |\psi_n(\mathbf{Q})|\ge kM(\mathbf{Q})\}$ for $k\in\mathbb{N}$.
Writing $\mathcal{E}_k = \{\mathbf{Q}\in B\ ;\ \Big(\ln(1+\left|\frac{\psi_n(\mathbf{Q})}{M(\mathbf{Q})}\right|)\Big)^2 \ge (\ln(1+k))^2\}$ we get

$$\int_B M\left(\ln(1+\left|\frac{\psi_n}{M}\right|)\right)^2 = \int_{\mathcal{E}_k} M\left(\ln(1+\left|\frac{\psi_n}{M}\right|)\right)^2 + \int_{B\setminus\mathcal{E}_k} M\left(\ln(1+\left|\frac{\psi_n}{M}\right|)\right)^2.$$

[3] We also use the Cauchy-Schwarz inequality to show that $\int_B Mf \le \sqrt{\int_B M}\sqrt{\int_B Mf^2} = \sqrt{\int_B Mf^2}$.

We easily deduce the following estimate

$$\int_B M\left(\ln(1+\left|\frac{\psi_n}{M}\right|)\right)^2 \geq \int_{\mathcal{E}_k} M\left(\ln(1+\left|\frac{\psi_n}{M}\right|)\right)^2 \geq \int_{\mathcal{E}_k} M(\ln(1+k))^2.$$

Introducing the measure $\mathrm{d}\mu = M(\mathbf{Q})\mathrm{d}\mathbf{Q}$, this inequality is rewritten too

$$\mu(\mathcal{E}_k) \leq \frac{1}{(\ln(1+k))^2}\left\|M\ln\left(1+\left|\frac{\psi_n}{M}\right|\right)\right\|_0^2$$

what, taking into account the estimate (26), reads

$$\mu(\{\mathbf{Q}\in B\ ;\ |\psi_n(\mathbf{Q})| \geq kM(\mathbf{Q})\}) \leq \frac{C}{(\ln(1+k))^2}. \tag{27}$$

Estimate of $\boldsymbol{MS_k\left(\frac{\psi_n}{M}\right)}$ in $\boldsymbol{H^1_{M,0}}$-norm - Recall that for $k\in\mathbb{N}$ the application T_k is given by $T_k : r \in \mathbb{R} \mapsto \max(\min(r,k),-k) \in \mathbb{R}$. We now define the application S_k such that $T_k + S_k = \mathrm{id}$. To obtain estimate on ψ_n we successively obtain estimate on $MS_k\left(\frac{\psi_n}{M}\right)$ and then on $MT_k\left(\frac{\psi_n}{M}\right)$ for sufficient large $k\in\mathbb{N}$.

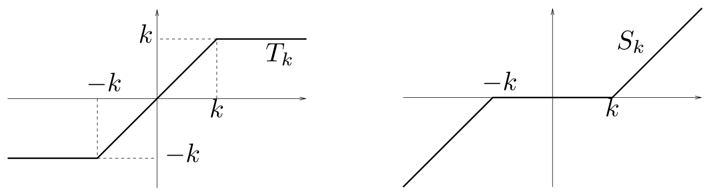


Figure 3: The functions T_k and S_k, $k\in\mathbb{N}^*$.

Let $k\in\mathbb{N}$. Taking $\varphi = MS_k\left(\frac{\psi_n}{M}\right) - M\int_B MS_k\left(\frac{\psi_n}{M}\right)$ as test function test in (22). According to lemma 3.3, this choice is possible and we have

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi_n}{M}\right)\cdot\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big) - \int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big) = \langle f,\varphi\rangle \tag{28}$$

Since $S_k + T_k = \mathrm{id}$ and for all $r\in\mathbb{R}$ we have $S_k'(r) = 0$ or $T_k'(r) = 0$ we deduce that the first term reads

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi_n}{M}\right)\cdot\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big) = \frac{1}{2\mathcal{D}e}\int_B M\left|\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big)\right|^2 = \frac{1}{2\mathcal{D}e}\left\|MS_k\left(\frac{\psi_n}{M}\right)\right\|_{1,0}^2. \tag{29}$$

Using the fact that for all $r\in\mathbb{R}$ we have $|T_n(r)| \leq |r|$ and using the Cauchy-Schwarz inequality, we estimate the second term in the following way

$$\left|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big)\right| \leq \|\boldsymbol{\kappa}\|_\infty\sqrt{\int_B\frac{|\psi_n|^2}{M}}\sqrt{\int_B M\left|\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big)\right|^2}.$$

However $\left|\frac{\psi_n}{M}\right| = \left|T_k\left(\frac{\psi_n}{M}\right) + S_k\left(\frac{\psi_n}{M}\right)\right| \leq k + \left|S_k\left(\frac{\psi_n}{M}\right)\right|$ thus $\left|\frac{\psi_n}{\sqrt{M}}\right| \leq k\sqrt{M} + \sqrt{M}\left|S_k\left(\frac{\psi_n}{M}\right)\right|$ and using the triangular inequality we get

$$\sqrt{\int_B\frac{|\psi_n|^2}{M}} \leq \sqrt{\int_B k^2M} + \sqrt{\int_B M\left|S_k\left(\frac{\psi_n}{M}\right)\right|^2} = k + \left\|MS_k\left(\frac{\psi_n}{M}\right)\right\|_0.$$

Since $S_k(r) = 0$ for $|r| < k$, we can estimate this last term as follow:

$$\left\|MS_k\left(\frac{\psi_n}{M}\right)\right\|_0^2 = \int_B M\left|S_k\left(\frac{\psi_n}{M}\right)\right|^2 = \int_{\mathcal{E}_k} M\left|S_k\left(\frac{\psi_n}{M}\right)\right|^2,$$

where we recall that $\mathcal{E}_k = \{\mathbf{Q} \in B \; ; \; |\psi_n(\mathbf{Q})| \geq kM(\mathbf{Q})\}$. According to the Hölder inequality, for all $p \in \mathbb{N}^*$, denoting by q the conjugate of p (i.e. such that $\frac{1}{p} + \frac{1}{q} = 1$) and using the estimate (27), we have

$$\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_0^2 \leq \Big(\int_{\mathcal{E}_k} M\Big)^{1/q}\Big(\int_{\mathcal{E}_k} M\Big|S_k\left(\frac{\psi_n}{M}\right)\Big|^{2p}\Big)^{1/p} \leq \frac{C}{(\ln(1+k))^{2/q}}\Big(\int_B M\Big|S_k\left(\frac{\psi_n}{M}\right)\Big|^{2p}\Big)^{1/p}.$$

We thus control the L^2_M-norm of $MS_k\left(\frac{\psi_n}{M}\right)$ using his L^p_M-norm . But this L^p_M-norms can itself be controlled, for adapted value of p by the H^1_M-norm. Indeed, using the continuous weighted Sobolev embedding (see lemma 3.6) there exists $p \in \mathbb{N}^*$ for which we have the inequality

$$\Big(\int_B M\Big|S_k\left(\frac{\psi_n}{M}\right)\Big|^{2p}\Big)^{1/p} \leq C\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_1^2 \leq C\Big(\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}^2 + \Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_0^2\Big).$$

We deduce a control on the L^2_M-norm of $MS_k\left(\frac{\psi_n}{M}\right)$ using his $H^1_{M,0}$-norm:

$$\Big(1 - \frac{C}{(\ln(1+k))^{2/q}}\Big)\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_0^2 \leq \frac{C}{(\ln(1+k))^{2/q}}\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}^2,$$

that is a control on the form $\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_0 \leq A(k)\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}$ where $\lim\limits_{k\to+\infty} A(k) = 0$. Hence, we get the following estimate for the second term of the left hand side of equation (28):

$$\Big|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(S_k\left(\frac{\psi_n}{M}\right)\Big)\Big| \leq \|\boldsymbol{\kappa}\|_\infty\Big(k + A(k)\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}\Big)\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}. \tag{30}$$

The last term of the equation (28) is controlled as follow

$$|\langle f, \varphi\rangle| \leq C\|\varphi\|_{1,0} = C\sqrt{\int_B M\Big|\nabla\left(\frac{\varphi}{M}\right)\Big|^2} \leq C\sqrt{\int_B M\Big|\nabla\Big(S_k\left(\frac{\varphi}{M}\right)\Big)\Big|^2} = C\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0} \tag{31}$$

The preceding estimates (29), (30) and (31) permit to deduce, from equation (28), for all $k \in \mathbb{N}$, the inequality

$$\frac{1}{2\mathcal{D}e}\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0} \leq \|\boldsymbol{\kappa}\|_\infty\Big(k + A(k)\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}\Big) + C.$$

Since $\lim\limits_{k\to+\infty} A(k) = 0$, it possible to obtain for k rather large the inequality (recall that all the constant named C do not depend on n)

$$\Big\|MS_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0} \leq C. \tag{32}$$

Estimate of $\boldsymbol{MT_k}\left(\frac{\boldsymbol{\psi_n}}{\boldsymbol{M}}\right)$ in $\boldsymbol{H^1_{M,0}}$-norm - Choose now $\varphi = MT_k\left(\frac{\psi_n}{M}\right) - M\int_B MT_k\left(\frac{\psi_n}{M}\right)$ as test function in equation (22) (according to lemma 3.3 we have $\varphi \in H^1_{M,0}$). As for the estimate of $MS_k\left(\frac{\psi_n}{M}\right)$, we study each of three terms present in equation (22). The first reads

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi_n}{M}\right)\cdot\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big) = \frac{1}{2\mathcal{D}e}\int_B M\Big|\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\Big|^2 = \frac{1}{2\mathcal{D}e}\Big\|MT_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}^2. \tag{33}$$

For the second term, we proceed as follow:

$$\Big|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\Big| \leq \|\boldsymbol{\kappa}\|_\infty\int_B M\Big|T_n\Big(\frac{\psi_n}{M}\Big)\Big|\Big|\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\Big| \leq \|\boldsymbol{\kappa}\|_\infty\int_B |\psi_n|\Big|\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\Big|.$$

But for $\left|\frac{\psi_n}{M}\right| \geq k$ we have $\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big) = 0$ whereas for $\left|\frac{\psi_n}{M}\right| < k$ we clearly have $|\psi_n| < kM$ and consequently, according to the Hölder inequality we get

$$\left|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\right| \leq \|\boldsymbol{\kappa}\|_\infty \int_B kM|\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)| \leq \|\boldsymbol{\kappa}\|_\infty\sqrt{\int_B k^2M}\sqrt{\int_B M\Big|\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\Big|^2}$$

Since $\int_B M = 1$ we obtain the following relation

$$\left|\int_B MT_n\Big(\frac{\psi_n}{M}\Big)\boldsymbol{\kappa}\cdot\nabla\Big(T_k\left(\frac{\psi_n}{M}\right)\Big)\right| \leq k\|\boldsymbol{\kappa}\|_\infty\Big\|MT_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}. \tag{34}$$

As for the last term it is treated like those of the preceding estimates:

$$\left|\langle f, MT_k\left(\frac{\psi_n}{M}\right)\rangle\right| \leq C\Big\|MT_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0}. \tag{35}$$

These estimates (33), (34) and (35) give

$$\Big\|MT_k\left(\frac{\psi_n}{M}\right)\Big\|_{1,0} \leq C. \tag{36}$$

Estimate of ψ_n in $H^1_{M,0}$ - Since for all $k \in \mathbb{N}$ we have $S_k + T_k = \mathrm{id}$ we get

$$\|\psi_n\|_{1,0} = \left\|MS_k\left(\frac{\psi_n}{M}\right) + MT_k\left(\frac{\psi_n}{M}\right)\right\|_{1,0} \leq \left\|MS_k\left(\frac{\psi_n}{M}\right)\right\|_{1,0} + \left\|MT_k\left(\frac{\psi_n}{M}\right)\right\|_{1,0}.$$

Using the estimates (32) and (36) we deduce that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\|\psi_n\|_{1,0} \leq C. \tag{37}$$

Convergence of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ - According to the estimate (37), the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{M,0}$. According to the lemma (3.5), a subsequence of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ (always denoted $\{\psi_n\}_{n\in\mathbb{N}}$) admits a limit ψ weak in $H^1_{M,0}$ and strong in L^2_M. In order to perform the limit in equation (22), it is enough to prove that the sequence $\{MT_n\left(\frac{\psi_n}{M}\right)\}_{n\in\mathbb{N}}$ goes to ψ in L^2_M. We get

$$\Big\|MT_n\left(\frac{\psi_n}{M}\right) - \psi\Big\|_0^2 \leq \Big\|MT_n\left(\frac{\psi_n}{M}\right) - MT_n\left(\frac{\psi}{M}\right)\Big\|_0^2 + \Big\|MT_n\left(\frac{\psi}{M}\right) - \psi\Big\|_0^2.$$

However the application $T : \mathbb{R} \to \mathbb{R}$ is 1-lipschitz and we have

$$\Big\|MT_n\left(\frac{\psi_n}{M}\right) - MT_n\left(\frac{\psi}{M}\right)\Big\|_0^2 = \int_B M\left|T_n\left(\frac{\psi_n}{M}\right) - T_n\left(\frac{\psi}{M}\right)\right|^2 \leq \int_B M\left|\frac{\psi_n}{M} - \frac{\psi}{M}\right|^2 = \|\psi_n - \psi\|_0^2$$

what proves that $\Big\|MT_n\left(\frac{\psi_n}{M}\right) - MT_n\left(\frac{\psi}{M}\right)\Big\|_0$ tends to 0 when n tends to $+\infty$. As regards the other term, the Lebesgue convergence dominated theorem directly affirms that $\Big\|MT_n\left(\frac{\psi}{M}\right) - \psi\Big\|_0$ also tends to 0 when n tends to $+\infty$. Finally, it was shown that the sequence $\{MT_n\left(\frac{\psi_n}{M}\right)\}_{n\in\mathbb{N}}$ converges to ψ in L^2_M and consequently that ψ is solution of

$$\int_B M\nabla\left(\frac{\psi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right) - \int_B \psi\boldsymbol{\kappa}\cdot\nabla\left(\frac{\varphi}{M}\right) = \langle f, \varphi\rangle \qquad \forall\varphi \in H^1_{M,0}.$$

□

4.3 Uniqueness proof in theorem 4.1

Main steps for the uniqueness proof - To prove uniqueness, we proceed as follow: We start by introducing the dual problem. It is shown that this dual problem admits a solution by using the Schauder topological degree method. Then, by using the existence both problem and its dual, we deduce uniqueness from these two problems.

Introduction of the dual problem - For $g \in H_M^{-1}$ let us consider the elliptic partial differential equation

$$-\frac{1}{2\mathcal{D}e}\,\mathrm{div}\left(M\nabla\left(\frac{\phi}{M}\right)\right) - M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\phi}{M}\right) = g \quad \text{on } B \tag{38}$$

and we look for a solution to this equation satisfying $\int_B \phi = \rho$ where ρ is a given real number.

Remark 4.2 *In equation* (19) *we have considered convection terms only in conservative form; in the dual equation* (38)*, we consider convection terms only in non-conservative form. It is important to note that we can't consider in the same equation convection terms both in conservative form and non-conservative form. Indeed a sum of a first order term under conservative form and a first order term under non-conservative form can create a zeroth order term (for instance $div(\psi\boldsymbol{\kappa}) - \boldsymbol{\kappa}\cdot\nabla\psi = (\,div\,\boldsymbol{\kappa})\psi$) and it is known that an equation of kind $-\Delta\psi + \lambda\psi = f$ can have no solution as soon as λ is an eigenvalue of the operator $-\Delta$ and $f = 0$.*

A compact application for the dual problem - For $\widetilde{\phi} \in H^1_{M,0}$ we have $M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}}{M}\right) \in L^2_M \subset H_M^{-1}$ since $\|M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}}{M}\right)\|_0 \le \|\boldsymbol{\kappa}\|_\infty\|\widetilde{\phi}\|_{1,0}$. There exists thus[4] a unique solution $\phi = G(\widetilde{\phi}) \in H^1_{M,0}$ to

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\phi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right) - \int_B \varphi\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}}{M}\right) = \langle g,\varphi\rangle \qquad \text{for all } \varphi \in H^1_{M,0}. \tag{39}$$

This defines an application $G : H^1_{M,0} \to H^1_{M,0}$. It is quite easy to see that G is continuous; indeed, if $\widetilde{\phi}_n$ goes to $\widetilde{\phi}$ in $H^1_{M,0}$ then $M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}_n}{M}\right)$ goes to $M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}}{M}\right)$ in H_M^{-1} (more precisely in L^2_M). Thus $\mathrm{div}\left(M\nabla\left(\frac{\phi_n}{M}\right)\right)$ goes to $\mathrm{div}\left(M\nabla\left(\frac{\phi}{M}\right)\right)$ which implies[4] that $\phi_n = G(\widetilde{\phi}_n)$ goes to $\phi = G(\widetilde{\phi})$ in $H^1_{M,0}$.

We will now prove that G is a compact operator. Suppose that the sequence $\{\widetilde{\phi}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{M,0}$; then $\{M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}_n}{M}\right)\}_{n\in\mathbb{N}}$ is bounded in H_M^{-1} so that, using $\varphi = G(\widetilde{\phi}_n) = \phi_n$ as test function in the equation satisfied by ϕ_n, we get using the lemma 3.4

$$\|\phi_n\|^2_{1,0} \le \Big(C + 2\mathcal{D}e\left\|M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}_n}{M}\right)\right\|_{H_M^{-1}}\Big)\|\widetilde{\phi}_n\|_{1,0},$$

which implies that the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{M,0}$. Using the lemma 3.5, up to a subsequence, we can thus suppose that $\{\phi_n\}_{n\in\mathbb{N}}$ converges a.e. on B and is bounded in L^2_M. Let $(n,m) \in \mathbb{N}^2$; subtract the equation satisfied by ϕ_m to the equation satisfied by ϕ_n and use $\varphi = \phi_n - \phi_m$ as test function, this gives using the lemma 3.4 again

$$\|\phi_n - \phi_m\|^2_{1,0} \le 2\mathcal{D}e\left|\int_B (\phi_n - \phi_m)\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}_n - \widetilde{\phi}_m}{M}\right)\right| \le C\|\phi_n - \phi_m\|_0.$$

From the strong convergence of $\{\phi_n\}_{n\in\mathbb{N}}$ to ϕ in L^2_M we deduce that the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{M,0}$ and converges in this space. We deduce that the application G is compact.

[4]We use the fact that the operator $\varphi \mapsto -\,\mathrm{div}\left(M\nabla\left(\frac{\varphi}{M}\right)\right)$ is coercitiv in $H^1_{M,0}$, that is the fact that the equation

$$-\,\mathrm{div}\left(M\nabla\left(\frac{\psi}{M}\right)\right) = f$$

has a solution in $H^1_{M,0}$ as soon as $f \in H_M^{-1}$, and $\|\psi\|_1 \le C\|f\|_{H_M^{-1}}$ where C is constant depending only on the domain B.

Existence result for the dual problem using the Leray-Schauder topological degree - We gives here only the points which are useful for us concerning the topological degree method. For a definition of the topological degree and for the principal properties that it checks, one will consult the founder article of J. Leray and J. Schauder [22].

Lemma 4.1 *Let E be a Banach space and $\mathcal{A}$ be the set of triplets $(\mathrm{Id}-G,\Omega,z)$ such that Ω is a bounded open in E, $z\in E$ and $G:\overline{\Omega}\to E$ a compact application with $z\notin(\mathrm{Id}-G)(\partial\Omega)$. There exists an application $d:\mathcal{A}\to\mathbb{Z}$ such that*

- *if $z\in\Omega$ then $d(\mathrm{Id},\Omega,z)=1$;*
- *if for all $s\in[0,1]$ we have $0\notin(\mathrm{Id}-sG)(\partial\Omega)$ then $d(\mathrm{Id},\Omega,0)=d(\mathrm{Id}-G,\Omega,0)$;*
- *If $d(\mathrm{Id}-G,\Omega,z)\neq 0$ then there exists $w\in\Omega$ such that $w-G(w)=z$.*

Remark 4.3 *Generally, to show that a compact operator G admits a fixed-point, since the last point of the lemma 4.1, it is sufficient to prove that $d(\mathrm{Id}-G,\Omega,0)\neq 0$. But using the two first points of this lemma 4.1, if we show that for $s\in[0,1]$ we have $0\notin(\mathrm{Id}-sG)(\partial\Omega)$ then we will get $d(\mathrm{Id}-G,\Omega,0)=d(\mathrm{Id},\Omega,0)=1\neq 0$ as soon as $0\in\Omega$ (what will be the case for example when Ω is a ball centered in 0).*

According to this remark, since the operator G introduced with equation (39) is a compact operator, to prove that it has a fixed point, we just have to find $R>0$ such that for all $s\in[0,1]$ there exists no solution of $\phi-sG(\phi)=0$ satisfying $\|\phi\|_{1,0}=R$. Let $s\in[0,1]$ and suppose that $\phi\in H^1_{M,0}$ satisfies $\phi=sG(\phi)$. We have

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\phi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right)-s\int_B\varphi\boldsymbol{\kappa}\cdot\nabla\left(\frac{\phi}{M}\right)=\langle sg,\varphi\rangle\qquad\text{for all }\varphi\in H^1_{M,0}.\tag{40}$$

Using the "non-dual" problem (see the existence proof of theorem 4.1 where we obtain an existence solution to equation (22)), we know that for all $f\in H^{-1}_M$ there exists at least one solution $\psi\in H^1_{M,0}$ to

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right)-s\int_B\psi\boldsymbol{\kappa}\cdot\nabla\left(\frac{\varphi}{M}\right)=\langle f,\varphi\rangle\qquad\text{for all }\varphi\in H^1_{M,0}.\tag{41}$$

Moreover, according to estimate (37) there exist $C_1\in\mathbb{R}^+$ such that for all $f\in H^{-1}_M$ with $\|f\|_{H^{-1}_M}\leq 1$ and for all $s\in[0,1]$ we have $\|\psi\|_{1,0}\leq C_1$. We can verify that this constant C_1 depends only on $\|f\|_{H^{-1}_M}$ and can be selected independently on the function f when $\|f\|_{H^{-1}_M}\leq 1$. In addition according to the estimates obtained in the existence proof of theorem 4.1 this constant C_1 depends on $\|s\boldsymbol{\kappa}\|_\infty$ but since $s\in[0,1]$ we have $\|s\boldsymbol{\kappa}\|_\infty\leq\|\boldsymbol{\kappa}\|_\infty$ and consequently the constant C_1 can also be selected independently of s.
By taking $\varphi=\phi$ in the equation (41) satisfied by ψ and $\varphi=\psi$ in the equation (40) satisfied by ϕ, we get

$$\langle f,\phi\rangle=\langle sg,\psi\rangle\leq s\|g\|_{H^{-1}_M}C_1\leq\|g\|_{H^{-1}_M}C_1:=C_2.$$

Since this inequality is satisfied for all $f\in H^{-1}_M$ such that $\|f\|_{H^{-1}_M}\leq 1$, we deduce that $\|\phi\|_{1,0}\leq C_2$.
Now take $R=C_2+1$. We have just proven that, for any $s\in[0,1]$, any solution to $\phi-sG(\phi)=0$ satisfies $\|\phi\|_{1,0}<R$; thus by the Leray-Schauder topological degree theory, the application G has a fixed point, that is to say a solution of (39).

Uniqueness - Since the equation (20) is linear, it is sufficient to prove that the only solution to (20) without source term, i.e. taking $f=0$, is the null function. Let ψ be a solution to (20) with $f=0$ and let ϕ a solution of (38) with[5] $g=\mathrm{sgn}(\psi)\in H^{-1}_M$. By putting $\varphi=\phi$ as test function in the equation (20) satisfied by ψ and

[5] For each $\psi\in H^1_M$, the function $\mathrm{sgn}(\psi)$ is defined as follow : for all $\varphi\in H^1_M$

$$\langle\mathrm{sgn}(\psi),\varphi\rangle=\int_{\{\mathbf{Q}\in B\ ;\ \psi(\mathbf{Q})>0\}}\varphi-\int_{\{\mathbf{Q}\in B\ ;\ \psi(\mathbf{Q})<0\}}\varphi.$$

We verify that this linear form on H^1_M is continuous since thanks to the Cauchy-Schwarz inequality we get

$$|\langle\mathrm{sgn}(\psi),\varphi\rangle|\leq 2\int_B|\varphi|\leq 2\sqrt{\int_B\frac{|\varphi|^2}{M}}\sqrt{\int_B M}=2\|\varphi\|_0\leq 2\|\varphi\|_{1,0}.$$

$\varphi = \psi$ as test function in the weak formulation of the equation (38) satisfied by ϕ, we respectively get

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi}{M}\right)\cdot\nabla\left(\frac{\phi}{M}\right) - \int_B \psi\boldsymbol{\kappa}\cdot\nabla\left(\frac{\phi}{M}\right) = 0 \qquad \text{and}$$
$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\phi}{M}\right)\cdot\nabla\left(\frac{\psi}{M}\right) - \int_B \psi\boldsymbol{\kappa}\cdot\nabla\left(\frac{\phi}{M}\right) = \langle \operatorname{sgn}(\psi), \psi\rangle.$$

We deduce that $\langle \operatorname{sgn}(\psi), \psi\rangle = 0$, that is to say $\int_B |\psi| = 0$ and then $\psi = 0$. □

Remark 4.4 *A similar reasoning gives the uniqueness of the solution to the dual problem* (39).

5 Non stationary solution

5.1 Existence result for a simplified equivalent problem

Let $\mathbf{u} \in \mathcal{C}(0, +\infty; W^{1,\infty}(\Omega))$ be a velocity vector field on a bounded open subset Ω of $\mathbb{R}^d$ without normal component on $\partial\Omega$, $\mathcal{D}e$ be the Deborah number quantifying the elasticity of the polymer, $\mathbf{F}$ be the spring elastic force of the FENE polymer model, that is $\mathbf{F}(\mathbf{Q}) = \frac{1}{1-\|\mathbf{Q}\|^2/\delta^2}$ for $\mathbf{Q} \in B$ where B is the open ball $B(0,\delta) \subset \mathbb{R}^d$ and δ corresponds to the maximal dumbbell elongation, and $\psi_{\text{init}} \in L^2_M$ be the initial distribution of the dumbbells. We show in this part that there exists an unique solution ψ depending on time $t \in \mathbb{R}^+$, on the macroscopic variable $\mathbf{x} \in \Omega$ and on the microscopic variable $\mathbf{Q} \in B$ to the following Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi = -\operatorname{div}_{\mathbf{Q}}\left((\nabla_{\mathbf{x}}\mathbf{u})^T\cdot\mathbf{Q}\psi - \frac{1}{2\mathcal{D}e}\mathbf{F}(\mathbf{Q})\psi - \frac{1}{2\mathcal{D}e}\nabla_{\mathbf{Q}}\psi\right) \tag{42}$$

such that the initial condition coïncides with ψ_{init}. Using the part 2, equation (42) can be rewritten as

$$\frac{\partial \psi}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi = -\operatorname{div}_{\mathbf{Q}}(\psi\boldsymbol{\kappa}) + \frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) \tag{43}$$

where M is the Maxwellian on B defined by the relation (12) and $\boldsymbol{\kappa}$ corresponds to the velocity influence, that is in classical case $\boldsymbol{\kappa}(t,\mathbf{x},\mathbf{Q}) = (\nabla_{\mathbf{x}}\mathbf{u}(t,\mathbf{x}))^T\cdot\mathbf{Q}$.
Since derivation with respect to the macroscopic variable $\mathbf{x}$ intervenes only in the convective terms, namely $\mathbf{u}\cdot\nabla_{\mathbf{x}}\psi$, we can start by treating the case of the parabolic equation with the scalar unknown ψ only depending on $t \in \mathbb{R}^*_+$ and on $\mathbf{Q} \in B$:

$$\frac{\partial \psi}{\partial t} = -\operatorname{div}_{\mathbf{Q}}(\psi\overline{\boldsymbol{\kappa}}) + \frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) \tag{44}$$

with $\psi(0,\mathbf{Q}) = \psi_{\text{init}}(\mathbf{Q})$ for all $\mathbf{Q} \in B$.
Indeed, if for each $\mathbf{X} \in \Omega$ we find a solution $(t,\mathbf{Q}) \mapsto \psi(t,\mathbf{X},\mathbf{Q})$ to equation (44) with $\overline{\boldsymbol{\kappa}}(t,\mathbf{Q}) = \boldsymbol{\kappa}(t,\mathbf{X},\mathbf{Q}) \in \mathcal{C}(0,+\infty; L^\infty(B))$ (the variable $\mathbf{X}$ is consider as parameter) then the function $\psi(t,\mathbf{x},\mathbf{Q})$ is solution of the system (43) where the relation between $\mathbf{X}$ and $\mathbf{x}$ is given by the following lemma (see [8]):

Lemma 5.1 *Let $\mathbf{u} \in \mathcal{C}([0,T]; W^{\alpha,p}(\Omega,\mathbb{R}^N))$ with $\alpha > \frac{N}{p}+1$ and $p \in \mathbb{N}^*$ such that $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Then the system*

$$\frac{d\mathbf{X}}{dt}(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{X}(t,\mathbf{x})) \quad \text{and} \quad \mathbf{X}(0,\mathbf{x}) = \mathbf{x}$$

has a solution $\mathbf{X} \in \mathcal{C}^1([0,T]; \mathcal{D}^{\alpha,p}(\Omega,\mathbb{R}^N))$ where $\mathcal{D}^{\alpha,p}(\Omega,\mathbb{R}^N)$ is the following space

$$\mathcal{D}^{\alpha,p}(\Omega,\mathbb{R}^N) = \{\zeta \in W^{\alpha,p}(\Omega,\mathbb{R}^N),\ \zeta \text{ is a bijection from } \overline{\Omega} \text{ to } \overline{\Omega} \text{ and } \zeta^{-1} \in W^{\alpha,p}(\Omega,\mathbb{R}^N)\}.$$

Thus, as for the stationary problem previously studied, we don't explicitly denote in this part the dependences in the variable $\mathbf{Q}$. Except for the time derivative which are explicitly indicate, all the derivatives (for instance ∇ or div) will be derivatives with respect to $\mathbf{Q}$.

The weak formulation of this equation (44) writes: find $\psi \in \mathcal{C}(0,+\infty;H^1_M)$ such that for all $\varphi \in H^1_M$

$$\frac{\partial}{\partial t}\left(\int_B \psi\frac{\varphi}{M}\right) + \frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right) = \int_B \psi\overline{\boldsymbol{\kappa}}\cdot\nabla\left(\frac{\varphi}{M}\right) \quad \text{in } \mathcal{D}'(0,+\infty). \tag{45}$$

We prove in the next part the following theorem

Theorem 5.1 *Let $B = B(0,\delta)$ with $\delta > \sqrt{2}$, $\overline{\boldsymbol{\kappa}} \in \mathcal{C}(0,+\infty;L^\infty(B,\mathbb{R}^d))$ and $M \in \mathcal{C}^\infty(\overline{B},\mathbb{R})$ be a normalized Maxwellian[6]. For all $\psi_{init} \in L^2_M$ there exists an unique solution $\psi \in \mathcal{C}(0,+\infty;L^2_M)\cap L^2_{loc}(0,+\infty;H^1_M)$ to equation (45) such that $\psi(0,\mathbf{Q}) = \psi_{init}(\mathbf{Q})$ for all $\mathbf{Q}\in B$. Moreover the $\mathbf{Q}$-average $\int_B \psi(t,\mathbf{Q})d\mathbf{Q}$ doesn't depend on time and*

- *if $\psi_{init}(\mathbf{Q}) \geq 0$ for all $\mathbf{Q}\in B$ then $\psi(t,\mathbf{Q}) \geq 0$ for all $(t,\mathbf{Q}) \in [0,+\infty[\times B$;*
- *if $\int_B \psi_{init} = 0$ and if $2\mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|_{\mathcal{C}(0,+\infty;L^\infty(B))} < 1$ then $\lim\limits_{t\to+\infty}\psi(t,\mathbf{Q}) = 0$ for all $\mathbf{Q}\in B$ (with exponential decreasing).*

5.2 Proof of theorem 5.1

Ideas for the existence proof - Concerning equation (45), there exists a simple a priori estimate (see estimate (46) below). To prove the existence of a solution to equation (45) it suffices to work with an approach problem on which such estimate holds. The lemma 3.1 permit us to use a Galerkin approximation ψ_n based on the eigenfunctions of the operator $\mathcal{L}$ (see [27] for the same method in a similar case). For clarity, we will present here the *a priori* estimates.

Average conservation - Taking $\varphi = M \in H^1_M$ as test function in the weak formulation (45) we get

$$\int_B \psi(t,\mathbf{Q})d\mathbf{Q} = \int_B \psi_{\text{init}}(\mathbf{Q})d\mathbf{Q} \qquad \forall t\in[0,T].$$

A priori estimate - Choosing $\varphi = \psi$ as test function in the weak formulation (45) we obtain[7]

$$\frac{d}{dt}\left(\frac{\|\psi\|_0^2}{2}\right) + \frac{1}{2\mathcal{D}e}\|\psi\|_{1,0}^2 = \int_B \psi\overline{\boldsymbol{\kappa}}\cdot\nabla\left(\frac{\psi}{M}\right).$$

Using the Cauchy-Schwarz inequality, we get

$$\frac{d}{dt}\left(\frac{\|\psi\|_0^2}{2}\right) + \frac{1}{2\mathcal{D}e}\|\psi\|_{1,0}^2 \leq \|\overline{\boldsymbol{\kappa}}\|_\infty\|\psi\|_0\|\psi\|_{1,0} \leq \frac{1}{4\mathcal{D}e}\|\psi\|_{1,0}^2 + \mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|_\infty^2\|\psi\|_0^2.$$

Using the Poincaré lemma (see lemma 3.4), we deduce that for all $\varepsilon > 0$ we have

$$\frac{d}{dt}\left(\frac{\|\psi\|_0^2}{2}\right) + \left(\frac{1}{4\mathcal{D}e} - \varepsilon\right)(\|\psi\|_0^2 - \rho_0^2) + \varepsilon\|\psi\|_{1,0}^2 - \mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|_\infty^2\|\psi\|_0^2 \leq 0.$$

We write this relation in the following form

$$\frac{d}{dt}\left(\frac{\|\psi\|_0^2}{2}\right) + \left(\frac{1}{4\mathcal{D}e} - 2\varepsilon - \mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|_\infty^2\right)\|\psi\|_0^2 + \varepsilon\|\psi\|_1^2 \leq \rho_0^2\left(\frac{1}{4\mathcal{D}e} - \varepsilon\right). \tag{46}$$

With this estimate, we easily deduce (after integrating with respect time and use the classical Gronwall lemma) that the sequence of approach solution (which come from the Galerkin method for instance) is bounded in $L^\infty(0,T;L^2_M)\cap L^2(0,T;H^1_M)$ for all $T\in\mathbb{R}_+^*$. To pass to the limit in equation (45), it suffices to find an

[6] That is to say that the function M satisfies $0 < M \leq 1$ on B, $M = 0$ on ∂B and $\int_B M = 1$

[7] We take care to the fact that $\|\cdot\|_{1,0}^2$ is not a norm on the space H^1_M but only a semi-norm.

estimate on $\frac{\partial \psi}{\partial t}$. Using the estimate (46), we know that $M\nabla\left(\frac{\psi}{M}\right) - M\overline{\boldsymbol{\kappa}}\cdot\nabla\left(\frac{\psi}{M}\right)$ is bounded in $L^2(0,T;L^2_M)$. Since $\frac{\partial \psi}{\partial t} = \frac{1}{2\mathcal{D}e}\,\mathrm{div}\left(M\nabla\left(\frac{\psi}{M}\right) - M\overline{\boldsymbol{\kappa}}\cdot\nabla\left(\frac{\psi}{M}\right)\right)$ we obtain

$$\left\|\frac{\partial \psi}{\partial t}\right\|_{L^2(0,T;H^{-1}_M)} \le C. \tag{47}$$

Existence result - The convergence of the Galerkin approximation sequence $\{\psi_n\}_{n\in\mathbb{N}}$ toward an application ψ solution of the problem (45) results from the estimates (46) and (47):

$$\begin{aligned}
&\psi_n \rightharpoonup \psi \quad \text{in } L^\infty(0,T;L^2_M) \text{ weak-}\star,\\
&\psi_n \rightharpoonup \psi \quad \text{in } L^2(0,T;H^1_M) \text{ weakly},\\
&\frac{\partial \psi_n}{\partial t} \rightharpoonup \frac{\partial \psi}{\partial t} \quad \text{in } L^2(0,T;H^{-1}_M) \text{ weakly}.
\end{aligned}$$

Uniqueness - Let ψ_1 and ψ_2 be two solutions of (45). Due to the linearity, the difference $\psi_2-\psi_1$ is solution of the same problem with zero initial condition (that is in particular with zero average: $\rho_0=0$). The estimate (46) allow us to obtain the following relation on $y=\|\psi_2-\psi_1\|_0^2$:

$$y'(t) \le Cy(t) \quad \text{on } \mathbb{R}^+.$$

Using the Gronwall lemma, we deduce that $y(t) \le e^{Ct}y(0)$. Since $y(0)=0$ we conclude that $y=0$ and consequently that $\psi_1=\psi_2$. That proves the uniqueness to the problem (45).

Long time behavior - Assume that $\int_B \psi_{\text{init}} = 0$, that is $\rho_0=0$. The energy estimate (46) reads

$$y'(t) + h(t)y(t) \le 0 \quad \text{on } \mathbb{R}^+,$$

where the function y corresponding to $y(t)=\|\psi\|_0^2(t)$ and the function h is defined by

$$h(t) = \frac{1}{4\mathcal{D}e} - \varepsilon - \mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|_\infty^2(t).$$

According to a Gronwall lemma, we have for all $t\in\mathbb{R}^+$

$$y(t) \le y(0)\exp\left(-\int_0^t h(s)\mathrm{d}s\right) = y(0)\exp\left(-\Big(\frac{t}{4\mathcal{D}e} - t\varepsilon - \mathcal{D}e\int_0^t \|\overline{\boldsymbol{\kappa}}\|_\infty^2(s)\mathrm{d}s\Big)\right). \tag{48}$$

To ensure the stability of the solution, it suffices that the quantity $\frac{t}{4\mathcal{D}e} - t\varepsilon - \mathcal{D}e\int_0^t \|\overline{\boldsymbol{\kappa}}\|_\infty^2$ tends to $+\infty$ when t tends to $+\infty$. If $\overline{\boldsymbol{\kappa}}\in\mathcal{C}(0,+\infty;L^\infty(B))$ then we have

$$\frac{t}{4\mathcal{D}e} - t\varepsilon - \mathcal{D}e\int_0^t \|\overline{\boldsymbol{\kappa}}\|_\infty^2 \ge t\left(\frac{1}{4\mathcal{D}e} - \varepsilon - \mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|^2_{\mathcal{C}(0,+\infty;L^\infty(B))}\right).$$

Under the assumption $2\mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|_{\mathcal{C}(0,+\infty;L^\infty(B))} < 1$ it is possible to choose $\varepsilon>0$ such that

$$\frac{1}{4\mathcal{D}e} - \varepsilon - \mathcal{D}e\|\overline{\boldsymbol{\kappa}}\|^2_{\mathcal{C}(0,+\infty;L^\infty(B))} > 0$$

and consequently such that $y(t)$ tends to 0 when t tends to $+\infty$.

Positivity - Taking $\varphi=\psi^-$ (the negative part of ψ) as test function in the weak formulation (45). This choice is licit since M being positive we have $\psi^- = M\vartheta\left(\frac{\psi}{M}\right)\in H^1_M$ where the application $\vartheta : r\in\mathbb{R}\mapsto \max(-r,0)\in\mathbb{R}$ is continuous, piecewise-$\mathcal{C}^1$ such that ϑ' is bounded on $\mathbb{R}$ (see lemma 3.3). We obtain

$$\frac{d}{dt}\left(\frac{\|\psi^-\|_0^2}{2}\right) + \frac{1}{2\mathcal{D}e}\int_B M\left|\nabla\left(\frac{\psi}{M}\right)\right|^2 = \int_B M\psi^-\overline{\boldsymbol{\kappa}}\cdot\nabla\left(\frac{\psi^-}{M}\right)$$

Like obtaining the estimate (46), we use the Cauchy-Schwarz inequality:

$$\frac{d}{dt}\left(\frac{\|\psi^-\|_0^2}{2}\right)+\frac{1}{2\mathcal{D}e}\|\psi^-\|_{1,0}^2\le\frac{\mathcal{D}e}{2}\|\overline{\boldsymbol{\kappa}}\|_\infty^2\|\psi^-\|_0^2+\frac{1}{2\mathcal{D}e}\|\psi^-\|_{1,0}^2,$$

so that the application z defined on $\mathbb{R}_+^*$ by $z=\|\psi^-\|_0^2$ satisfies $z'\le C\|\overline{\boldsymbol{\kappa}}\|_\infty^2 z$. Using the Gronwall lemma, if $\psi_{\text{init}}\ge 0$, that is if $y(0)=0$ then $y(t)=0$ for all t, that proves that $\psi^-=0$. We deduce that $\psi\ge 0$. □

Remark 5.1
- If $\overline{\boldsymbol{\kappa}}\in L^2(0,+\infty;L^\infty(B))$, choosing $\varepsilon<\frac{1}{4\mathcal{D}e}$ in the estimate (48), we clearly have the stability result, that is $y(t)$ tends to 0 when t tends to $+\infty$, without other smallness conditions about $\mathcal{D}e$ or $\overline{\boldsymbol{\kappa}}$.
- More generally, the optimal condition on $\mathcal{D}e$ and $\overline{\boldsymbol{\kappa}}$ to obtain a stability result from estimate (48) reads

$$4\mathcal{D}e^2\lim_{t\to\infty}\frac{1}{t}\int_0^t\|\overline{\boldsymbol{\kappa}}\|_\infty^2<1.$$

5.3 Existence result for the stress contribution in the FENE model

Using the theorem 5.1 we can deduce existence result, uniqueness result and asymptotic time behavior for the FENE model describe by equation (42). Indeed, for $\mathbf{x}\in\Omega$ let $\overline{\boldsymbol{\kappa}}(t,\mathbf{Q})=(\nabla_{\mathbf{x}}\mathbf{u}(t,\mathbf{x}))^T\cdot\mathbf{Q}$. Assuming that $\mathbf{u}\in\mathcal{C}(0,+\infty;W^{1,\infty}(\Omega))$ we have get $\overline{\boldsymbol{\kappa}}\in\mathcal{C}(0,+\infty;L^\infty(B))$ with

$$\|\overline{\boldsymbol{\kappa}}\|_{\mathcal{C}(0,+\infty;L^\infty(B))}=\delta\|\mathbf{u}\|_{\mathcal{C}(0,+\infty;W^{1,\infty}(\Omega))}.$$

Hence, the lemma 5.1 coupled with the theorem 5.1 gives

Corollary 5.1 *Let $B=B(0,\delta)$ with $\delta>\sqrt{2}$, $\mathbf{u}\in\mathcal{C}(0,+\infty;W^{1,\infty}(\Omega))$ and $M\in\mathcal{C}^\infty(\overline{B},\mathbb{R})$ be a normalized Maxwellian[8]. For all $\psi_{init}\in L^\infty(\Omega)\otimes L_M^2$ there exists an unique weak solution $\psi\in\mathcal{C}(0,+\infty;L^\infty(\Omega)\otimes L_M^2)\cap L_{loc}^2(0,+\infty;L^\infty(\Omega)\otimes H_M^1)$ to equation (42) such that $\psi(0,\mathbf{x},\mathbf{Q})=\psi_{init}(\mathbf{x},\mathbf{Q})$ for all $(\mathbf{x},\mathbf{Q})\in\Omega\times B$. Moreover the mean value $\int_B\psi(t,\mathbf{x},\mathbf{Q})d\mathbf{Q}$ doesn't depend on time and*

- *if $\psi_{init}(\mathbf{x},\mathbf{Q})\ge 0$ for all $(\mathbf{x},\mathbf{Q})\in\Omega\times B$ then $\psi(t,\mathbf{x},\mathbf{Q})\ge 0$ for all $(t,\mathbf{x},\mathbf{Q})\in[0,+\infty[\times\Omega\times B$;*
- *if $\int_B\psi_{init}(t,\mathbf{x},\mathbf{Q})\mathrm{d}\mathbf{Q}=0$ for all $(t,\mathbf{x})\in[0,+\infty[\times\Omega$ and if we have $2\delta\mathcal{D}e\|\mathbf{u}\|_{\mathcal{C}(0,+\infty;W^{1,\infty}(\Omega))}<1$ then we get $\lim\limits_{t\to+\infty}\psi(t,\mathbf{x},\mathbf{Q})=0$ for all $(\mathbf{x},\mathbf{Q})\in\Omega\times B$ (with exponential decreasing).*

Using this corollary, we deduce that the solution to the non-stationary Fokker-Planck equation (7) tends to the solution to the stationary equation (6) when the time t tends to $+\infty$ as soon as the velocity is small enough.

Indeed, let $\psi(t,\mathbf{x},\mathbf{Q})$ be the solution of (7) with $\psi_{\text{init}}(\mathbf{x},\mathbf{Q})$ as initial condition. Consider the solution $\widetilde{\psi}(\mathbf{x},\mathbf{Q})$ of equation (6) with $\int_B\widetilde{\psi}(\mathbf{x},\mathbf{Q})d\mathbf{Q}=\int_B\psi_{\text{init}}(\mathbf{x},\mathbf{Q})d\mathbf{Q}$. It is easy to see that $\psi(t,\mathbf{x},\mathbf{Q})-\widetilde{\psi}(\mathbf{x},\mathbf{Q})$ is a solution of equation (7) with zero $\mathbf{Q}$-average. Hence, according to corollary 5.1 if $2\delta\mathcal{D}e\|\mathbf{u}\|_{\mathcal{C}(0,+\infty;W^{1,\infty}(\Omega))}<1$ then $\psi(t,\mathbf{x},\mathbf{Q})-\widetilde{\psi}(\mathbf{x},\mathbf{Q})$ tends to 0 as t tends to $+\infty$. That is $\psi(t,\mathbf{x},\mathbf{Q})\to\widetilde{\psi}(\mathbf{x},\mathbf{Q})$ when $t\to+\infty$.

For instance, for a co-rotationel flow we explicitely know the stationary solution of equation (6), see section 2.3. Hence if the velocity is small enough then the solution of the Fokker-Planck equation with ψ_{init} as initial condition tends to $\psi(t,\mathbf{x},\mathbf{Q})\to\left(\int_B\psi_{\text{init}}(\mathbf{x},\mathbf{Q})d\mathbf{Q}\right)M$ when $t\to+\infty$.

Remark 5.2
- The existence result obtained in corollary 5.1 was already shown (see for instance [23, Lemma 4] or [34, Lemma 3]). This corollary not only makes it possible to prove the existence but also to understand the long time behavior of solutions.
- The condition $2\delta\mathcal{D}e\|\mathbf{u}\|_{\mathcal{C}(0,+\infty;W^{1,\infty}(\Omega))}<1$ corresponds to an excess of energy: this excess is compensed in a complete model (taking to account the momentum equation and not only the constitutive law for the constraint), see for instance [1], [23] or [27]. Nevertheless, for the next part and applications developed in this paper, the interesting case corresponds to the case where $\mathbf{u}$ is small.

[8]That is to say that the function M satisfies $0<M\le 1$ on B, $M=0$ on ∂B and $\int_B M=1$

According to the corollary 5.1 and using the lemma 2.1 we can deduce the following result.

Corollary 5.2 *If the velocity field satisfies $\mathbf{u} \in \mathcal{C}(0,+\infty; W^{1,\infty}(\Omega))$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ then the FENE model is well posed in the sense that it correspond a unique polymer stress which satisfies $\boldsymbol{\sigma}_P \in \mathcal{C}(0,+\infty; L^\infty(\Omega))$.*

Proof: By definition (see equation (15)), we have $\boldsymbol{\sigma}_P = \lambda\left(\langle \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\rangle - \rho \mathbf{Id}\right)$. It is obvious that if $\psi \in \mathcal{C}(0,+\infty; L^\infty(\Omega) \otimes L^2_M))$ then $\rho = \int_B \psi \in \mathcal{C}(0,+\infty; L^\infty(\Omega))$ (we use here the fact that $L^2_M \subset L^2(B) \subset L^1(B)$). Next, we have

$$\langle \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\rangle = \int_B \sqrt{M(\mathbf{Q})}\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\frac{\psi(\mathbf{Q})}{\sqrt{M(\mathbf{Q})}}d\mathbf{Q}.$$

Using the Cauchy-Schwarz inequality, we get

$$|\langle \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\rangle| \leq \|\psi\|_0 \sqrt{\int_B M(\mathbf{Q})|\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}|^2 d\mathbf{Q}}$$

where we recall that the norm $|\cdot|$ denote the maximal component of a tensor: $|\mathbf{A}| = \sup_{i,j}|A_{i,j}|$. Using the lemma 2.1, we deduce that $\langle \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\rangle \in \mathcal{C}(0,+\infty; L^\infty(\Omega))$ and consequently that $\boldsymbol{\sigma}_P \in \mathcal{C}(0,+\infty; L^\infty(\Omega))$. $\square$

6 Asymptotic behavior and time boundary layer

According to the previous part (section 5), we know that for a given function $\psi_{\text{init}} \in L^\infty(\Omega) \otimes L^2_M$ and for each $\varepsilon > 0$ the following system admits an unique solution ψ^ε depending on $(t, \mathbf{x}, \mathbf{Q}) \in \mathbb{R}_+ \times \Omega \times B$

$$\varepsilon\left(\frac{\partial \psi^\varepsilon}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi^\varepsilon\right) - \frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi^\varepsilon}{M(\mathbf{Q})}\right)\right) + \operatorname{div}_{\mathbf{Q}}\left(\psi^\varepsilon\left(\boldsymbol{\kappa} + \varepsilon\widetilde{\boldsymbol{\kappa}}\right)\right) = 0 \tag{49}$$

such that $\psi^\varepsilon(0, \mathbf{x}, \mathbf{Q}) = \psi_{\text{init}}(\mathbf{x}, \mathbf{Q})$, and according to part 4, the following system (formally obtained by taking $\varepsilon = 0$ in the preceding one) admits a unique solution ψ^0 depending on $(\mathbf{x}, \mathbf{Q}) \in \Omega \times B$

$$-\frac{1}{2\mathcal{D}e}\operatorname{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi^0}{M(\mathbf{Q})}\right)\right) + \operatorname{div}_{\mathbf{Q}}\left(\psi^0\boldsymbol{\kappa}\right) = 0 \tag{50}$$

and such that $\int_B \psi^0(\mathbf{x}, \mathbf{Q})d\mathbf{Q} = \int_B \psi_{\text{init}}(\mathbf{x}, \mathbf{Q})d\mathbf{Q}$.

6.1 Main results

We rigorously justifie the convergence of the solution ψ^ε to ψ^0 when ε goes to 0. More precisely, we show the following result:

Theorem 6.1 *Let $B = B(0,\delta)$ be a ball of $\mathbb{R}^d$ of radius $\delta > \sqrt{2}$, $\boldsymbol{\kappa} \in L^\infty(\Omega) \otimes L^\infty(B)$, $\widetilde{\boldsymbol{\kappa}} \in \mathcal{C}(0,+\infty; L^\infty(\Omega) \otimes L^\infty(B))$, $M \in \mathcal{C}^\infty(\overline{B}, \mathbb{R})$ be a normalized Maxwellian*[9]*, $\mathbf{u} \in \mathcal{C}(0,+\infty; W^{1,\infty}(\Omega))$ and $\psi_{init} \in L^\infty(\Omega) \otimes L^2_M$. For each $\varepsilon \in \mathbb{R}^*_+$ we denote by $\psi^\varepsilon \in \mathcal{C}(0,+\infty; L^\infty(\Omega) \otimes L^2_M) \cap L^2_{loc}(0,+\infty; L^\infty(\Omega) \otimes H^1_M)$ the solution of equation (49) and by $\psi^0 \in L^\infty(\Omega) \otimes H^1_M$ the solution of equation (50). Then there exists two functions $\widetilde{\psi^0}$ and Ψ in $\mathcal{C}(0,+\infty; L^\infty(\Omega) \otimes L^2_M) \cap L^2_{loc}(0,+\infty; L^\infty(\Omega) \otimes H^1_M)$ such that*

$$\psi^\varepsilon(t, \mathbf{x}, \mathbf{Q}) = \psi^0(\mathbf{x}, \mathbf{Q}) + \widetilde{\psi^0}\left(\frac{t}{\varepsilon}, \mathbf{x}, \mathbf{Q}\right) + \varepsilon\Psi(t, \mathbf{x}, \mathbf{Q}).$$

The function $\widetilde{\psi^0}$ is called a time boundary layer. For small values of $\boldsymbol{\kappa}$, it satisfies $\lim_{\tau\to+\infty}\widetilde{\psi^0}(\tau, \mathbf{x}, \mathbf{Q}) = 0$ (with exponential decreasing). Moreover, if $\psi_{init} = \psi^0$ then $\widetilde{\psi^0} = 0$.

[9]That is to say that the function M satisfies $0 < M \leq 1$ on B, $M = 0$ on ∂B and $\int_B M = 1$

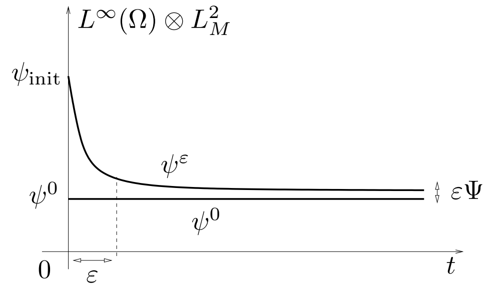


Figure 4: Illustration of the theorem 6.1.

Remark 6.1

- When $\psi_{init} = \psi^0$, that is when the initial condition ψ_{init} of the systems (49)*, for all $\varepsilon > 0$, coincides with the solution ψ^0 of the stationary problem* (50)*, we say that data are well-prepared. In the contrary case, we say that data are ill-prepared (see [10]).*

- We deduce from this theorem that ψ^ε tends to ψ^0 in $L^2([0,+\infty[; L^2(\Omega)\otimes H^1_M)$ and that the convergence takes place in $L^\infty([0,+\infty[; L^2(\Omega)\otimes H^1_M)$ when data are well-prepared.

- More generally, we can show (exactly as in theorem 6.1) that for each $N \in \mathbb{N}$, there exist some functions ψ^0, ..., ψ^N, some profiles $\widetilde{\psi^0}$, ..., $\widetilde{\psi^N}$ and a remainder Ψ such that

$$\psi^\varepsilon(t,\mathbf{x},\mathbf{Q}) = \psi^0(\mathbf{x},\mathbf{Q}) + \widetilde{\psi^0}\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \cdots + \varepsilon^N\psi^N(\mathbf{x},\mathbf{Q}) + \varepsilon^N\widetilde{\psi^N}\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \varepsilon^{N+1}\Psi(t,\mathbf{x},\mathbf{Q}).$$

If $\boldsymbol{\kappa}$ is small enough the functions $\widetilde{\psi^k}$ introduced above satisfy $\lim\limits_{\tau\to+\infty}\widetilde{\psi^k}(\tau,\mathbf{x},\mathbf{Q}) = 0$ (with exponential decreasing) and the remainder is bounded independently of ε.

6.2 Proof of theorem 6.1

The proof is organized in three steps. The first consists in building an approximate solution: we carry out a formal asymptotic extension of the solution. In the second step, we solve the profile equations: the first one corresponding to the initial equations without the term ε, the second one to an equation in which it is necessary to control the decay in the fast variable. The third step consists in showing that the remainder of the extension is bounded in an adequate space.

Boundary layer profile - We seek an asymptotic extension of ψ^ε in the form

$$\psi^\varepsilon(t,\mathbf{x},\mathbf{Q}) = \psi_0\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \varepsilon\psi_1\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \varepsilon^2\psi_2\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \cdots$$

For such a method, it is convenient to introduce the following notations. For all $k \in \mathbb{N}$, we have

$$\psi_k(\tau,\mathbf{x},\mathbf{Q}) = \overline{\psi}_k(\mathbf{x},\mathbf{Q}) + \widetilde{\psi}_k(\tau,\mathbf{x},\mathbf{Q}),$$

where $\overline{\psi}_k(\mathbf{x},\mathbf{Q}) = \lim\limits_{\tau\to+\infty}\psi_k(\tau,\mathbf{x},\mathbf{Q})$ and $\widetilde{\psi}_k$ with fast decay in τ.
We then replace formally ψ^ε by its asymptotic extension in the equation (49). We then seek to determine the profile ψ_k by identifying all terms of the same order in ε.
At the order 0, we get

$$\frac{\partial\,\psi_0}{\partial\tau} - \frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi_0}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left(\psi_0\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q})\right) = 0 \tag{51}$$

and we impose $\psi_0(\mathbf{x},\mathbf{Q},0) = \psi_{\text{init}}(\mathbf{x},\mathbf{Q})$. We let $\tau\to+\infty$ and we deduce

$$-\frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\overline{\psi}_0}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left(\overline{\psi}_0\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q})\right) = 0 \tag{52}$$

where the $\mathbf{Q}$-average of $\overline{\psi}_0$ is given by $\int_B \overline{\psi}_0(\mathbf{x},\mathbf{Q})\,d\mathbf{Q} = \int_B \overline{\psi}_{\rm init}(\mathbf{x},\mathbf{Q})\,d\mathbf{Q}$. From the equation (51), we then obtain

$$\frac{\partial\,\widetilde{\psi}_0}{\partial\tau} - \frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\widetilde{\psi}_0}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left(\widetilde{\psi}_0\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q})\right) = 0 \tag{53}$$

with $\widetilde{\psi}_0(\mathbf{x},\mathbf{Q},0) = \psi_{\rm init}(\mathbf{x},\mathbf{Q}) - \overline{\psi}_0(\mathbf{x},\mathbf{Q})$. Using the same method, we get at order $k \geq 1$ the following equations for the profile $\overline{\psi}_k$ and $\widetilde{\psi}_k$:

$$-\frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\overline{\psi}_k}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left(\overline{\psi}_k\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q})\right) = -\,\mathrm{div}_{\mathbf{Q}}\left(\overline{\psi}_{k-1}\widetilde{\boldsymbol{\kappa}}(\mathbf{x},\mathbf{Q})\right) - \mathbf{u}\cdot\nabla_{\mathbf{x}}\overline{\psi}_{k-1} \tag{54}$$

with zero $\mathbf{Q}$-average and

$$\frac{\partial\,\widetilde{\psi}_k}{\partial\tau} - \frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\widetilde{\psi}_k}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left(\widetilde{\psi}_k\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q})\right) = -\,\mathrm{div}_{\mathbf{Q}}\left(\widetilde{\psi}_{k-1}\widetilde{\boldsymbol{\kappa}}(\mathbf{x},\mathbf{Q})\right) - \mathbf{u}\cdot\nabla_{\mathbf{x}}\widetilde{\psi}_{k-1} \tag{55}$$

with zero initial value.

Asymptotic extension - In the study we have just undertaken, we obtained the main term of the extension $\psi_0 = \overline{\psi}_0 + \widetilde{\psi}_0$ where $\overline{\psi}_0$ and $\widetilde{\psi}_0$ are solutions respectively of the problem (52) and (53). These two systems were studied previously (see Theorem 4.1 for the solution to equation (52) and Corollary 5.1 for the solution to equation (53)) what makes it possible to affirm the existence of each profile. On the same way, using previous parts, we can prove that each profile ψ_k is well defined as the sum of the solutions of (54) and (55).

Convergence of the extension - The study previously carried out is only formal and to justify that the development of ψ in power of ε is rigorously a development (that is converges) we write ψ^ε in the following form

$$\psi^\varepsilon(t,\mathbf{x},\mathbf{Q}) = \psi_0\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \varepsilon\Psi(t,\mathbf{x},\mathbf{Q}) \tag{56}$$

where ψ_0 is the profile determine above and we prove that the remainder Ψ is bounded. Clearly, to obtain a rigorous development until the order $k \geq 1$, we have

$$\psi^\varepsilon(t,\mathbf{x},\mathbf{Q}) = \psi_0\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \cdots + \varepsilon^k\psi_k\Big(\frac{t}{\varepsilon},\mathbf{x},\mathbf{Q}\Big) + \varepsilon^{k+1}\Psi(t,\mathbf{x},\mathbf{Q})$$

where ψ_i $(0 \leq i \leq k)$ are the profiles determine above and we prove that the remainder Ψ is bounded. For sake of simplicity, we show here the case of the zeroth order.
Introduce the profile given by equation (56) in the equation (49). Using the equations satisfying by $\overline{\psi}_0$ and $\widetilde{\psi}_0$ (that is by ψ_0) the following equation on the remainder Ψ is obtained

$$\varepsilon\frac{\partial\,\Psi}{\partial t} - \frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\Psi}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left(\Psi(\boldsymbol{\kappa}(\mathbf{x},\mathbf{Q}) + \varepsilon\widetilde{\boldsymbol{\kappa}}(\mathbf{x},\mathbf{Q}))\right) = -\,\mathrm{div}_{\mathbf{Q}}\left(\psi_0\widetilde{\boldsymbol{\kappa}}(\mathbf{x},\mathbf{Q})\right) - \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi_0 \tag{57}$$

with zero initial value: $\widetilde{\psi}_0(\mathbf{x},\mathbf{Q},0) = 0$. Thus the equation satisfy by $\widetilde{\Psi}(\tau,\mathbf{x},\mathbf{Q}) = \Psi(t,\mathbf{x},\mathbf{Q})$ where $t = \varepsilon\tau$ is the same one as that obtained for Ψ (equation (57)) except that we replace $\varepsilon\frac{\partial\,\Psi}{\partial t}$ by $\frac{\partial\,\widetilde{\Psi}}{\partial\tau}$.
According to part 5, we have the following estimate

$$\|\widetilde{\Psi}\|_{\mathcal{C}(0,+\infty;L^2_M)\cap L^2_{\rm loc}(0,+\infty;H^1_M)} \leq C,$$

where C does not depend on ε. We deduce that the remainder Ψ satisfy

$$\|\Psi\|_{\mathcal{C}(0,+\infty;L^2_M)} \leq C \quad \text{and} \quad \|\Psi\|_{L^2_{\rm loc}(0,+\infty;H^1_M)} \leq \varepsilon C,$$

what concludes this demonstration. $\square$

7 Applications to viscoelastic laminar boundary layers and lubrication problems

7.1 Anisotropic flows

In many natural flows or in laboratory, we know that one of the direction of the flow is privileged. It is for example the case when the geometry in which the fluid moves is "anisotropic". Thus, if $\Omega = [0,L]\times[0,H] \subset \mathbb{R}^2$ with $H \ll L$ then it is natural to distinguish in the non-dimensional step the two characteristic lengths $L_\star$ and $H_\star$, then revealing the ratio

$$\varepsilon := \frac{H_\star}{L_\star} \ll 1.$$

For such flows, it is usual to distinguish from the same manner the horizontal velocity u of the fluid and its vertical velocity v. It is generally supposed that two associated characteristic velocities $U_\star$ and $V_\star$ satisfy the relation $V_\star = \varepsilon U_\star$. This choice makes it possible to preserve in non-dimensional form the free-divergence relation $\operatorname{div}(\mathbf{u}) = 0$.
In such a domain, the velocity gradient write in a non-dimension form

$$\nabla_{\mathbf{x}}\mathbf{u} = \begin{pmatrix} \partial_x u & \dfrac{1}{\varepsilon}\partial_z u \\ \varepsilon\partial_x v & \partial_z v \end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix} 0 & \partial_z u \\ 0 & 0 \end{pmatrix} + \mathcal{O}(1).$$

Physically, this means that the flow is managed by a shear flow. It is thus natural to wonder whether the behavioral law of a fluid can be rigorously approximate by a simpler law in an anisotropic flow. For instance, in a flow of Newtonian fluid, the constraint is given by the relation $\boldsymbol{\sigma} = \eta\left(\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T\right)$ which is written, in the case of an anisotropic flow describes above:

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \eta\partial_z u \\ \eta\partial_z u & 0 \end{pmatrix} + \mathcal{O}(\varepsilon).$$

7.2 FENE model for thin flows

Let us consider a polymer whose behavior of the stress is given by the FENE model, i.e. such that the relation between the constraint and velocity obeys the relations (10) and (15). Let us suppose that the flow is anisotropic as defined in the preceding paragraph, so that the velocity field is written $\nabla_{\mathbf{x}}\mathbf{u} = \frac{1}{\varepsilon}\nabla_{\mathbf{x}}\mathbf{u}_0 + \mathcal{O}(1)$. Moreover, in order to observe the microscopic effects, let us assume that $\mathcal{D}e$ is of order of ε (for sake of simplicity, it signifies that we replace $\mathcal{D}e$ by $\varepsilon\mathcal{D}e$).

Remark 7.1 *About this choice for the size of the Deborah number $\mathcal{D}e$, notice that, concerning the Oldroyd model, the same remark is essential to be interested in the non common effects of elasticity (see [3] for more explanations). If the Deborah number $\mathcal{D}e$ is not correctly correlated with the small parameter ε then either the effects of elasticity are invisible (in the case where $\mathcal{D}e$ is too small with respect to ε) or the effects are translated by additional viscous contributions (in the case where $\mathcal{D}e$ is too large).*

In this case, denoting by ψ^ε the probability distribution function of the dumbbell orientation, the Fokker-Planck equation (10) corresponds to equation (49). We know according to the preceding parts, and in particular according to theorem 6.1, that the solution of this equation (49) behaves like the solution of equation (50) when ε becomes small.
We deduce that the polymeric contribution of the stress for an anisotropic flow is written, for all $(t,\mathbf{x}) \in \mathbb{R}_+\times\Omega$,

$$\boldsymbol{\sigma}_P(t,\mathbf{x}) = \boldsymbol{\sigma}_0(\mathbf{x}) + \mathcal{O}(\varepsilon)$$

where $\boldsymbol{\sigma}_0$ is given by the relation (15), in which ψ is the solution of the equation (50). It is consequently enough to solve this equation (50) to obtain an approximation at order 0 of the stress. Moreover, according to work of Bird and al. [7, Equation 13.5-15, p. 79] (see also the part 2.3 of this paper) a development of the solution ψ of the equation (50) for small Deborah numbers (or in other words close to an equilibrium state) is given by:

$$\begin{aligned}\psi(\mathbf{x},\mathbf{Q}) = \rho\, M(\mathbf{Q})\Bigg(1+\frac{\mathcal{D}e}{2}\mathbf{D}(\mathbf{x}) : \mathbf{Q}\otimes\mathbf{Q} + \frac{\mathcal{D}e^2}{4}\bigg(\frac{1}{2}(\mathbf{D}(\mathbf{x}) : \mathbf{Q}\otimes\mathbf{Q})^2 - \frac{1}{15}\langle\|\mathbf{Q}\|^4\rangle_{\mathrm{eq}}\mathbf{D}(\mathbf{x}) : \mathbf{D}(\mathbf{x}) \\ + \frac{4\delta^2}{2\delta^2+7}(1-\frac{\|\mathbf{Q}\|^2}{2\delta^2})(\mathbf{D}(\mathbf{x})\cdot\mathbf{W}(\mathbf{x})) : \mathbf{Q}\otimes\mathbf{Q}\bigg) + \mathcal{O}(\mathcal{D}e^3)\Bigg),\end{aligned} \tag{58}$$

where the notation $\langle\,\cdot\,\rangle_{\mathrm{eq}}$ corresponds to $\int_B \cdot\, \rho\, M(\mathbf{Q})\, d\mathbf{Q}$, and $\mathbf{D}(\mathbf{x})$ and $\mathbf{W}(\mathbf{x})$ are respectively the symmetric and skew-symmetric part of the velocity gradient $\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x})$.

Since the relation between the stress $\boldsymbol{\sigma}$ and the probability density ψ is a linear relation, we can directly deduce a development of $\boldsymbol{\sigma}$ using the development of ψ. We have

$$\sigma_0(\mathbf{x}) = \sigma^{\mathrm{eq}} + \mathcal{D}e\ \sigma^{(1)} + \mathcal{D}e^2\ \sigma^{(2)} + \mathcal{O}(\mathcal{D}e^3),$$

where each term can be determine using the relation (15) and the development (58) of ψ. For instance, the equilibrium contribution $\rho\, M(\mathbf{Q})$ provides the term of order 0 in the following way:

$$\sigma^{\mathrm{eq}} = \frac{\lambda\rho}{J}\left(\int_B \mathbf{Q}\otimes\mathbf{Q}\left(1-\frac{\|\mathbf{Q}\|^2}{\delta^2}\right)^{\frac{\delta^2}{2}-1} d\mathbf{Q}\right) - \lambda\rho\ \mathbf{Id}$$

where J is the normalisation constant given by formula (12) p. 5. In the 3-dimensional case, we explicit this contribution using spherical change of coordinates (see [12] for more details):

$$\begin{array}{rcl}]0,\delta[\times]0,\pi[\times]-\pi,\pi[& \longrightarrow & B \\ (r,\theta,\varphi) & \longmapsto & (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta), \end{array}$$

whose the jacobian is given by $\mathrm{Jac}(r,\theta,\varphi) = r^2\sin\theta$. We get the following form for the equilibrium contribution to the stress:

$$\sigma^{\mathrm{eq}} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -\lambda\rho \end{pmatrix} \quad \text{with } a = \frac{4\lambda\rho\pi\beta(4)}{3J} - \lambda\rho.$$

This expression makes appear the beta function defined by

$$\beta(q) = \int_0^\delta r^q\left(1-\frac{r^2}{\delta^2}\right)^{\frac{\delta^2}{2}-1} dr.$$

Remark 7.2

- *This function β can be defined using the classical Euler integrale of first kind:*

$$\beta(q) = \frac{1}{2}\delta^{(q-1)/2}\mathrm{Eul}\Big(\frac{q+1}{2},\frac{\delta^2}{2}\Big) \quad \textit{where} \quad \mathrm{Eul}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

- *In the same way, the constant J and the term $\langle\|\mathbf{Q}\|^4\rangle_{eq}$ appearing in equation (58) can be written using the Euler integrale:*

$$J = 2\pi\delta^3\mathrm{Eul}\Big(\frac{3}{2},\frac{\delta^2+2}{2}\Big) \quad \textit{and} \quad \langle\|\mathbf{Q}\|^4\rangle_{eq} = 4\pi\beta(6).$$

Thereafter we will see that an important case corresponds to the shear flow, i.e. when the tensor of the deformations takes the following form:

$$\mathbf{D} = \begin{pmatrix} 0 & \dot\gamma & 0 \\ \dot\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where the coefficient $\dot{\gamma}$ is called the shear rate. In that case, the expression of the stress is determined relatively easily (see [12]). We have

$$\sigma^{(1)} = \begin{pmatrix} 0 & b\,\dot{\gamma} & 0 \\ b\,\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^{(2)} = \begin{pmatrix} (c+d)\,\dot{\gamma}^2 & 0 & 0 \\ 0 & (c-d)\,\dot{\gamma}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the constant b, c and d are explicitely given with respect to the function β, the constant normalisation J and the physical constant λ, ρ and δ:

$$b = \frac{4\lambda\rho\pi\beta(6)}{15J}, \qquad c = \frac{\lambda\rho\pi}{315\,J}\Big(9\beta(8) - 56\pi\beta(4)\beta(6)\Big), \qquad d = \frac{4\lambda\rho\pi}{15\,J\,(2\delta+7)}\Big(2\delta\beta(4) - \beta(6)\Big).$$

Thus, the developments with order 2 of each non constant component of the stress for shear flow are written (only for the polymeric contribution):

$$\begin{cases} \sigma_0^{11} = a + (c+d)\,\mathcal{D}e^2\,\dot{\gamma}^2 + \mathcal{O}(\mathcal{D}e^3) \\ \sigma_0^{12} = b\,\mathcal{D}e\,\dot{\gamma} + \mathcal{O}(\mathcal{D}e^3) \\ \sigma_0^{22} = a + (c-d)\,\mathcal{D}e^2\,\dot{\gamma}^2 + \mathcal{O}(\mathcal{D}e^3) \end{cases} \tag{59}$$

Since $b \neq 0$ and $d \neq 0$, this law highlights the tangential stresses appearing at the order 1 as well as the normal efforts at the order 2. In other words, it is natural to propose as asymptotic model with the FENE model in thin flows, the following constitutive law:

$$\boldsymbol{\sigma}_P = a\,\mathbf{Id} + b\,\mathcal{D}e\,\mathbf{D} + c\,\mathcal{D}e^2\,\mathbf{D}^2 + d\,\mathcal{D}e^2\,\mathbf{A}\,\mathbf{D}^2 \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

7.3 Applications to viscoelastic laminar boundary layers

Boundary layer flow for non-Newtonian fluids has been studied in few cases: for a second grade fluid (see for instance [28]), for a Walter's B fluid in [5], for an Oldroyd-B fluid in [4] and more recently for a FENE-P fluid in [30, 31]. In virtue of what was presented previously, we are able to introduce a new model for the study of the boundary layers for a fluid FENE.

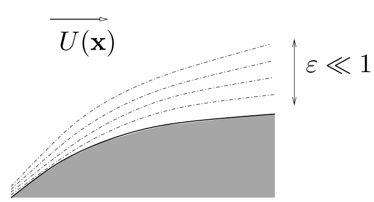


Figure 5: Boundary layer geometry.

More precisely, we interest in the case of plane flows in layers neighboring with a rigid wall. Considering flows in a thin layer of thickness of order $\varepsilon \ll 1$, the dynamic equation of equilibrium and the incompressibility condition can be written in a the non-dimension form (see for instance [43])

$$\begin{cases} u\partial_x u + v\partial_z u = -\partial_x p^* + \partial_z\sigma^{12} + \partial_x(\sigma^{11} - \sigma^{22}) + \mathcal{O}(\varepsilon) \\ \partial_z p^* = 0 + \mathcal{O}(\varepsilon) \\ \partial_x u + \partial_z v = 0 \end{cases} \tag{60}$$

In this model, which comes from to the classical conservations laws (see equations (1)) in thin domain, notice that the modified pressure p^* corresponds to $p^* = p - \sigma^{22}$. The concept of a viscoelastic boundary layer may be based on a rather intuitive than physical assumptions that in numerous practical situations there exists some sufficiently thin layer close to the wall in which viscoelastic effects are meaningful, and the outside flow is exactly an in-viscid one, governed by the Euler equations. Under these assumptions, the external flow is described by

$$d_x p^* = -U d_x U \tag{61}$$

where $U(x)$ denotes the velocity resulting from an in-viscid solution for $x > 0$ and $y = 0$.

Remark 7.3

- To obtain this kind of models, it is necessary to make assumptions on the characteristic size of the pressure p *and of the Reynolds number* $\mathcal{R}e$*. This non-dimensional number represents the relationship between the inertias and the viscous forces. It is defined by* $\mathcal{R}e = \frac{\mu U_\star L_\star}{\eta}$ *where* μ *is the density and* η *the viscosity of the fluid. More exactly, the model* (60) *is obtained when the pressure is of order* 1 *and that the Reynolds number is of order* $1/\varepsilon^2$.

- In the Newtonian fluid case, the system (60) *leads to the Prandtl equations. By adding the expression of the pressure* (61)*, we have*

$$\begin{cases} u\partial_x u + v\partial_z u = U d_x U + \dfrac{1}{\mathcal{R}e}\partial_z^2 u \\ \quad \partial_x u + \partial_z v = \ 0 \end{cases}$$

For this Newtonian model it was shown (see for instance [36]) that there exist self-similar solutions with these equations, i.e. solutions depending only on y *up to a change of variable.*

For a viscoelastic fluid describes by the micro-macro FENE model we know that the stress can be expressed, in the neighborhood of an equilibrium, by the relations (59). In particular the polymeric contribution gives

$$\sigma_0^{12} = b\,\mathcal{D}e\,\partial_z u + \mathcal{O}(\mathcal{D}e^3, \varepsilon) \quad \text{and} \quad \sigma_0^{11} - \sigma_0^{22} = 2d\,\mathcal{D}e^2\,(\partial_z u)^2 + \mathcal{O}(\mathcal{D}e^3, \varepsilon).$$

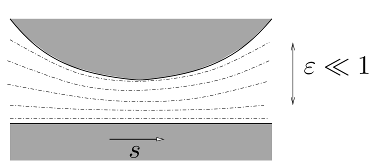
whereas the newtonian contribution reads $\sigma_N^{12} = \frac{1}{\mathcal{R}e}\partial_z u$ and $\sigma_N^{11} - \sigma_N^{22} = 0$. To give an account of these contributions in the boundary layer, it will thus be necessary to be interested in the following model:

$$\begin{cases} u\partial_x u + v\partial_z u = U d_x U + \Big(\dfrac{1}{\mathcal{R}e} + b\mathcal{D}e\Big)\partial_z^2 u + 2d\,\mathcal{D}e^2\,\partial_x((\partial_z u)^2) \\ \quad \partial_x u + \partial_z v = \ 0 \end{cases} \tag{62}$$

These governing equation being derivated, the possibility of self-similar solutions can be discussed. It is also interesting to understand the effect of the Deborah number on such flows. If it is clear that its effect at first order influences only viscosity, its effect at second order brings terms of normal forces which will have a considerable effects on the solutions. A theoretical and numerical work on this subject is currently in preparation, see [12].

7.4 Applications to lubrication problems

We presented in the preceding paragraph an example of anisotropic geometry. It is to be noticed that this kind of anisotropy is very usual in another domain of applications. This is the case in lubrication studies which are mainly devoted to thin film flow, in the study of the spreading of tears or in description of polymers through thin dies. In such domains, some particular classes of non-Newtonian fluids are often considered. This includes the Bingham flow or the quasi-Newtonian fluids, see [39]. In lubrication problems, the elastic character of a fluid seems to play a considerable part. In this framework, viscoelastic models of thin film fluids were already studied by J. Tichy [40] for the Maxwell model of viscoelasticity and by G. Bayada and al. [3] for models obeying a Oldroyd-B law.



Lubrication geometry.

More precisely, we interest in the two-dimensional case (the three dimensional case is similar) and consider flows in a thin layer of thickness $h(x)$ of order $\varepsilon \ll 1$. Moreover for application, one of the boundary has a nonnull velocity s (see previous figure). As in the preceding application developed in part 7.3, we introduce the pressure $p^* = p - \sigma^{22}$. The main difference with the previous model is the fact that in lubrication problems the flow is controlled more by the pressure forces than by turbulences. In other words, p^* is of order $1/\varepsilon^2$ whereas $\mathcal{R}e$ is of order 1. The dynamic equation of equilibrium and the incompressibility condition can be written in the non-dimension form

$$\begin{cases} -\partial_z^2 u + \partial_x p^* = \partial_z \sigma^{12} + \partial_x(\sigma^{11} - \sigma^{22}) + \mathcal{O}(\varepsilon) \\ \partial_z p^* = 0 + \mathcal{O}(\varepsilon) \\ \partial_x u + \partial_z v = 0 \end{cases} \tag{63}$$

Remark 7.4 *In the Newtonian fluid case, there is no polymer contribution in the stress. That is $\boldsymbol{\sigma} = \mathbf{0}$ in the model* (63)*. Integrating twice the first equation of* (63) *with respect to z we obtain the velocity u with respect the pressure p^*. Then, using the free-divergence condition as $\partial_x \left(\int_0^{h(x)} u(x,z)dz \right) = 0$, it possible in this case to deduce an equation on the pressure:*

$$\partial_x \Big(\frac{h^3}{12} \partial_x p^* \Big) = \partial_x \Big(\frac{h}{2\mathcal{R}e} s \Big).$$

This equation, known under the name of Reynolds equation, is a parabolic equation whose study is relatively simple (even in dimension 3). It was obtained in a heuristic way by O. Reynolds [35] then rigorously starting from the Stokes equations by G. Bayada and M. Chambat [2].

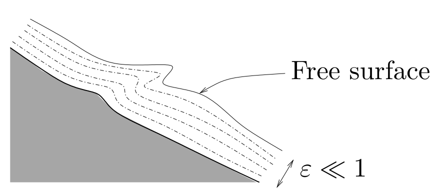
If the flow is a viscoelastic flow of FENE type, using the approximation (59) suggested for the anisotropic flows, the equation (63) writes at main order:

$$\begin{cases} -\Big(\frac{1}{\mathcal{R}e} + b\,\mathcal{D}e \Big) \partial_z^2 u + \partial_x p^* = 2d\,\mathcal{D}e^2\,\partial_x((\partial_z u)^2) \\ \partial_z p^* = 0 \\ \partial_x u + \partial_z v = 0 \end{cases}$$

The following questions are then natural: can we deduce, as in the Newtonian case, a generalized Reynolds equation on the pressure? Which are the effects of elasticity (i.e. the effect of the Deborah number $\mathcal{D}e$) on this model?

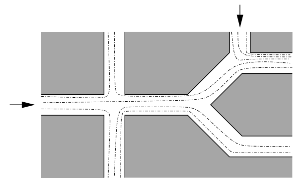
7.5 Other applications to free surface flows, to microfluidic, to Shallow-Water equations or fluid-structure interaction in biology?

In addition to the two preceding applications, we can use the reduced FENE model in many different physical contexts. Thus, the Shallow-water equations which describe a flow taking into account free surface can adapt to the cases of the viscoelastic fluids of FENE type: Materials involved in geophysical flows exhibit non-Newtonian rheological properties and, over the last few years, a great deal of work has been expended to adapt the shallow-water equations to non-Newtonian fluids. The long wave asymptotic usually used in these problems allows the use the model suggested in this paper.



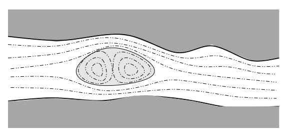
Application to non-newtonian Shallow-Water equations.

Many applications relate to the microfluidic industry. One of the objectives being to understand the flows in microchannels in order to be able to carry out mixtures and chemical reactions with very little of fluid. This type of problems enters completely within the framework of our study as soon as the concerned fluids have viscoelastic behavior. This type of approach can be adapted for example to interior stagnation point flows of viscoelastic liquids which arise in a wide variety of applications including extensional viscometry, polymer processing and microfluidics, see [42].



Application to microfluidic devices.

To finish, we can naturally think to biological application and in particular to blood circulations which have a viscoelastic behavior and which take place in arteries, typically anisotropic mediums where the model suggested would make it possible to “simply” understand the micro-macro effect. Two different approaches can be considered: either we study the cellular dynamics in the arteries, or we focus on the modelling of the fluid-structure interaction mechanism in vascular dynamics. See for instance [16].



Cellular dynamics in the arteries / fluid-structure interaction.

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